HIGH SCHOOL ALGEBRA

ADVANCED COURSE

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HIGH SCHOOL ALGEBRA

Advanced Course

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PREFACE

The Advanced Course of the High School Algebra contains a review of all topics treated in the Elementary Course, together with such additional topics as are required to make it amply sufficient to meet the entrance requirements of any college or technical school. Its development is based upon the following important considerations:

1. The pupil has had a one year's course in algebra, involving constant application of its elementary processes to the solution of concrete problems. This has invested the processes themselves with an interest which now makes them a proper object of study for their own sake.

2. The pupil has, moreover, developed in intellectual maturity and is, therefore, able to comprehend processes of reasoning with abstract numbers which were entirely beyond his reach in the first year's course. This is particularly true if, in the meantime, he has learned to reason with the more concrete forms of geometry.

In consequence of these considerations, the treatment throughout is from a more mature point of view than in the Elementary Course. The principles of algebra are given in the form of theorems the proofs of which are based upon a definite set of axioms.

As in the Elementary Course, the important principles are used at once in the solution of concrete and interesting problems, which, however, are here adapted to the pupil's greater maturity and experience. But relatively greater space and emphasis are given to the manipulation of standard algebraic

forms, such as the student is likely to meet in later work in mathematics and physics, and especially such as were too complicated for the Elementary Course.

The division of the High School Algebra into two distinct courses has made it possible to give in the Advanced Course a more thorough treatment of the elements of algebra than could be given if the book were designed for first-year classes. It has thus become possible to lay emphasis upon the pedagogic importance of viewing each subject a second time in a manner more profound than is possible on a first view.

Attention is specifically called to the following points:

The scientific treatment of axioms in Chapter I.

The clear and simple treatment of equivalent equations in Chapter III.

The discussion by formula, as well as by graph, of inconsistent and dependent systems of linear equations, pages 40 to 44.

The unusually complete treatment of factoring and the clear and simple exposition of the general process of finding the Highest Common Factor, in Chapter V.

The careful discrimination in stating and applying the theorems on powers and roots in Chapter VI.

The unique treatment of quadratic equations in Chapter VII, giving a lucid exposition in concrete and graphical form of distinct, coincident, and imaginary roots.

The concise treatment of radical expressions in Chapter X, and especially — an innovation much needed in this connection — the rich collection of problems, in the solution of which radicals are applied.

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CONTENTS

CHAPTER I

FUNDAMENTAL LAWS

	PAGE
Axioms of Addition and Subtraction	1
Axioms of Multiplication and Division	3
Theorems on Addition and Subtraction	6
Theorems on Multiplication and Division	8

CHAPTER II

FUNDAMENTAL OPERATIONS

Definitions	11
Addition and Subtraction of Monomials	12
Addition and Subtraction of Polynomials	15
Removal of Parentheses	17
Multiplication and Division of Monomials	18
Multiplication and Division of Polynomials	21

CHAPTER III

INTEGRAL EQUATIONS OF THE FIRST DEGREE IN ONE UNKNOWN

Definitions	25
Equivalent Equations	27
Problems in One Unknown	32

CHAPTER IV

INTEGRAL LINEAR EQUATIONS IN TWO OR MORE VARIABLES

Indeterminate Equations	36
Simultaneous Equations in Two Variables	38

PAGE
Inconsistent and Dependent Equations 43
Systems in More than Two Variables 45
Problems in Two or More Unknowns 47

CHAPTER V

FACTORING

Expression of Two, Three, or Four Terms 52
Factors found by Grouping 55
Factors found by the Factor Theorem 57
Solution of Equations by Factoring 60
Common Factors and Multiples 61

CHAPTER VI

POWERS AND ROOTS

Definitions 69
Theorems on Powers and Roots 73
Roots of Polynomials 77
Roots of Arabic Numbers 80

CHAPTER VII

QUADRATIC EQUATIONS

Exposition by means of Graphs 83
Distinct, Coincident, and Imaginary Roots 87
Simultaneous Quadratics 90
Special Methods of Solution 96
Higher Equations involving Quadratics 106
Relations between the Roots and Coefficients 108
Formation of Equations with Given Roots 110
Problems involving Quadratics 112

CHAPTER VIII

ALGEBRAIC FRACTIONS

Reduction of Fractions 116
Addition and Subtraction of Fractions 120

PAGE
Multiplication and Division of Fractions 123
Complex Fractions 126
Equations involving Algebraic Fractions 127
Problems involving Fractions 133

CHAPTER IX

RATIO, VARIATION, AND PROPORTION

Ratio and Variation 135
Proportion 139
Problems 141

CHAPTER X

EXPONENTS AND RADICALS

Fractional and Negative Exponents 142
Reduction of Radical Expressions 147
Addition and Subtraction of Radicals 150
Multiplication and Division of Radicals 151
Rationalizing the Divisor 155
Square Root of Radical Expressions 156
Equations containing Radicals 159
Problems involving Radicals 163

CHAPTER XI

LOGARITHMS

Definitions and Principles 167

CHAPTER XII

PROGRESSIONS

Arithmetic Progressions 175
Geometric Progressions 181
Harmonic Progressions 186

CHAPTER XIII

THE BINOMIAL FORMULA

Proof by Induction 189
The General Term 191

HIGH SCHOOL ALGEBRA

ADVANCED COURSE

CHAPTER I

FUNDAMENTAL LAWS

1. We have seen in the Elementary Course that **algebra**, like arithmetic, deals with **numbers** and with operations upon numbers. We now proceed to study in greater detail the laws that underlie these operations.

THE AXIOMS OF ADDITION AND SUBTRACTION

In adding numbers we assume at the outset certain **axioms**.

2. Axiom I. *Any two numbers have one and only one sum.*

Since two numbers are equal when and only when they are the same number, it follows from this axiom that if $a = b$ and $c = d$ then $a + c = b + d$.

For if a is the same number as b, and c is the same number as d, then adding b and d is the same as adding a and c, and by Axiom I the sums are the same and hence equal.

Therefore from Axiom I follows the axiom usually given: *If equal numbers be added to equal numbers, the sums are equal numbers.*

Since Axiom I asserts that the sum of two numbers is unique, it is often called the **uniqueness axiom of addition.**

3. If $a = c$ and $b = c$ then $a = b$, since the given equations assert that a is the same number as b. Hence the usual statement: *If each of two numbers is equal to the same number, they are equal to each other.*

4. The sum of two numbers, as 6 and 8, may be found by adding 6 to 8 or 8 to 6, in either case obtaining 14 as the result. This is a particular case of a general law for all numbers of algebra, which we enunciate as

Axiom II. *The sum of two numbers is the same in whatever order they are added.*

This is expressed in symbols by the identity:

$$a + b \equiv b + a. \qquad \text{[See § 37, E. C.*]}$$

Axiom II states what is called the **commutative law of addition,** since it asserts that numbers to be added may be *commuted* or interchanged in order.

Definition. Numbers which are to be added are called **addends.**

5. In adding three numbers such as 5, 6, and 7, we first add two of them and then add the third to this sum. It is immaterial whether we first add 5 and 6 and then add 7 to the sum, or first add 6 and 7 and then add 5 to the sum. This is a particular case of a general law for all numbers of algebra, which we enunciate as

Axiom III. *The sum of three numbers is the same in whatever manner they are grouped.*

In symbols we have

$$a + b + c \equiv a + (b + c).$$

When no symbols of grouping are used, we understand $a + b + c$ to mean that a and b are to be added first and then c is to be added to the sum.

Axiom III states what is called the **associative law of addition,** since it asserts that addends may be *associated* or grouped in any desired manner.

It is to be noted that an equality may be read in either direction. Thus $\quad a + b + c = a + (b + c)$ and $a + (b + c) = a + b + c$ are equivalent statements.

6. If any two numbers, such as 19 and 25, are given, then in arithmetic we can always find a number which added to

* E. C. means the Elementary Course.

the smaller gives the larger as a sum. That is, we can subtract the smaller number from the larger.

In Algebra, where negative numbers are used, any number may be subtracted from any other number. That is:

Axiom IV. *For any pair of numbers* a *and* b *there is one and only one number* c *such that* $a + c = b$.

The process of finding the number c when a and b are given is called **subtraction.** b is the **minuend**, a the **subtrahend**, and c the **remainder.** This operation is also indicated thus, $b - a = c$.

If $a + c = a$, then the number c is called **zero**, and is written 0. That is, $a + 0 = a$, or $a - a = 0$.

Adding a to each member of the equality $b - a = c$, we have $b - a + a = c + a$, which by hypothesis is equal to b. Hence *subtracting a number and then adding the same number gives as a result the original number operated upon.*

Axiom IV is called the **uniqueness axiom of subtraction.** A direct consequence is the following: *If equal numbers are subtracted from equal numbers, the remainders are equal numbers.*

THE AXIOMS OF MULTIPLICATION AND DIVISION

7. Axioms similar to those just given for addition and subtraction hold for multiplication and division.

Axiom V. *Two numbers have one and only one product.*

This is called the **uniqueness axiom of multiplication.** It is a direct consequence of this axiom that: *If equal numbers are multiplied by equal numbers, the products are equal numbers.*

8. The product of 5 and 6 may be obtained by taking 5 six times, or by taking 6 five times. That is, $5 \cdot 6 = 6 \cdot 5$. This is a special case of a general law for all numbers of algebra, which we enunciate as

Axiom VI. *The product of two numbers is the same in whatever order they are multiplied.*

In symbols we have $\quad a \cdot b = b \cdot a.$

This axiom states what is called the **commutative law of factors** in multiplication.

9. The product of three numbers, such as 5, 6, and 7, may be obtained by multiplying 5 and 6, and this product by 7, or 6 and 7, and this product by 5. This is a special case of a general law for all numbers of algebra, which we enunciate as

Axiom VII. *The product of three numbers is the same in whatever manner they are grouped.*

In symbols we have $abc = a(bc)$.

The expression abc without symbols of grouping is understood to mean that the product of a and b is to be multiplied by c.

This axiom states what is called the **associative law of factors** in multiplication.

Principles III and XV of E. C. follow from Axioms VI and VII.

10. Another law for all numbers of algebra is enunciated as

Axiom VIII. *The product of the sum or difference of two numbers and a given number is equal to the result obtained by multiplying each number separately by the given number and then adding or subtracting the products.*

In symbols we have

$$a(b + c) \equiv ab + ac \text{ and } a(b - c) \equiv ab - ac.$$

Axiom VIII states what is called the **distributive law of multiplication.**

When these identities are read from left to right, they are equivalent to Principle IV, E. C., and when read from right to left (see § 5) they are equivalent to Principles I and II, E. C. In the form $a(b \pm c) \equiv ab \pm ac$ this axiom is directly applicable to the multiplication of a polynomial by a monomial, and in the form $ab \pm ac = a(b \pm c)$, to the addition and subtraction of monomials having a common factor.

11. **Axiom IX.** *For any two numbers,* a *and* b*, provided* a *is not equal to zero,** *there is one and only one number* c *such that*

$$a \cdot c = b.$$

* The symbol for the expression *a is not equal to zero* is $a \neq 0$.

Definitions. If $ac = b$, the process of finding c when a and b are given is called **division.** b is the **dividend**, a the **divisor**, c the **quotient**, and we write $b \div a = c$, or $\frac{b}{a} = c$. For the case, $a = 0$, see §§ 24, 25.

If $a \cdot c = a$, $a \neq 0$, then the number c is called **unity**, and is written 1. That is, $\frac{a}{a} = 1$.

Multiplying both sides of the equality $\frac{b}{a} = c$ by a, we have $a \cdot \frac{b}{a} = ac$, which by hypothesis equals b. Hence *dividing by a number and then multiplying by the same number gives as a result the original number operated upon.*

Axiom IX is called the **uniqueness axiom of division.** As a direct consequence of this axiom we have: *If equal numbers are divided by equal numbers, the quotients are equal numbers.*

12. Axioms I, IV (in case the subtrahend is not greater than the minuend), V, and IX underlie respectively the processes of addition, subtraction, multiplication, and division, from the very beginning in elementary arithmetic. Axioms II, III, VI, VII, and VIII are also fundamental in arithmetic, where they are usually assumed without formal statement.

E.g. Axiom VIII is used in long multiplication such as 125×235, where we multiply 125 by 5, by 30, and by 200, and then add the products.

13. Negative Numbers. Axiom IV, in case the subtrahend is greater than the minuend, does not hold in arithmetic because of the absence of the negative number. This axiom therefore *brings the negative number into algebra.*

We now proceed to study the laws of operation upon this *enlarged number system.* In the Elementary Course concrete applications were used to show that certain rules of signs hold in operations upon positive and negative numbers. We shall now see that the same rules follow from the axioms just stated.

14. Definitions. If $a + b = 0$, then b is said to be the **negative** of a and a the negative of b. If a is a positive number, that is an ordinary number of arithmetic, then b is called a **negative number.** We denote the negative of a by $-a$. Hence, $a + (-a) = 0$. a and $-a$ have the same **absolute value.**

If $a - b$ is positive, then a is said to be *greater than* b. This is written $a > b$. If $a - b$ is negative, then a is said to be *less* than b. This is written $a < b$. If $a - b = 0$, then $a = b$, and if $a = b$ then $a - b = 0$. See § 6.

THEOREMS ON ADDITION AND SUBTRACTION

Definition. A **theorem** is a statement to be proved.

A **corollary** is a theorem which follows directly from some other theorem.

15. Theorem 1. *Adding a negative number is equivalent to subtracting a positive number having the same absolute value. That is,*

$$\boldsymbol{a + (-b) = a - b} \qquad \text{See § 48, E. C.}$$

Proof. Let $$a + (-b) = x. \tag{1}$$

Such a number x exists by Axiom I.

Adding b to each member of (1), $$a + (-b) + b = x + b. \tag{2}$$

By the associative law of addition, § 5, and by §§ 14, 6,

$$a + (-b) + b = a + [(-b) + b] = a + 0 = a. \tag{3}$$

From (2) and (3) by § 3, $$x + b = a. \tag{4}$$

From (4), by the definition of subtraction, § 6,

$$a - b = x. \tag{5}$$

From (1) and (5) by § 3, $$a + (-b) = a - b.$$

It follows from theorem **1** that either of the symbols, $+(-b)$ or $-b$, may replace the other in any algebraic expression.

16. Corollary. *A parenthesis preceded by the plus sign may be removed without changing the sign of any term within it.* See § 28, E. C.

For, since by the theorem $b - a = b + (-a)$, each subtraction is reducible to an addition, so that the associative law, § 5, applies. Thus

$$a + (b - c + d) = a + [b + (-c) + d] = a + b + (-c) + d = a + b - c + d.$$

Hence an expression may be inclosed in a parenthesis preceded by the plus sign without changing the sign of any of its terms.

17. Theorem 2. *Subtracting a negative number is equivalent to adding a positive number having the same absolute value. That is,*

$$\boldsymbol{a - (-b) \equiv a + b}. \qquad \text{See § 60, E. C.}$$

Proof. Let $a - (-b) = x$. (1)

From (1) by § 2, $a - (-b) + (-b) = x + (-b)$. (2)

From (2) by §§ 6 and 15, $a = x + (-b) = x - b$. (3)

Hence by the definition of subtraction, $a + b = x$. (4)

From (1) and (4) by § 3, $a - (-b) = a + b$. (5)

It follows from theorem **2** that either of the symbols $-(-b)$ or $+b$ may replace the other in any algebraic expression.

18. Theorem 3. *A parenthesis preceded by the minus sign may be removed by changing the sign of each term within it. That is,*

$$\boldsymbol{a - (b - c + d) \equiv a - b + c - d}. \qquad \text{See § 28, E. C.}$$

Proof. Let $a - (b - c + d) = x$. (1)

From (1) by the definition of subtraction,

$$a = x + (b - c + d). \qquad (2)$$

By §§ 15, 16, $a = x + b + (-c) + d$. (3)

Adding $(-b)$, c, and $(-d)$ to each member and using § 4,

$$a + (-b) + c + (-d) = x + b + (-b) + c + (-c) + d + (-d). \qquad (4)$$

From (4), by §§ 14, 15, $a - b + c - d = x$. (5)

From (1) and (5) by § 3, $a - (b - c + d) = a - b + c - d$. (6)

It follows from equation (6), read from right to left, that an expression may be inclosed in a parenthesis preceded by a minus sign, if the sign of each term within is changed.

19. Corollary 1. $\boldsymbol{a - b = -(b - a)}$.

For by §§ 15 and I, $a - b = a + (-b) = -b + a$. (1)

Hence by § 18, $a - b = -(b - a)$. (2)

20. Corollary 2. $\boldsymbol{-a + (-b) = -(a + b)}$. See § 48, E. C.

For by § 18, $-a + (-b) = -[a - (-b)]$. (1)

Hence by § 17, $-a + (-b) = -[a + b]$. (2)

21. From the identities

$$a + (-b) \equiv a - b, \qquad § 15,$$

$$a - (-b) \equiv a + b, \qquad § 17,$$

$$a - b \equiv -(b - a), \qquad § 19,$$

$$-a + (-b) \equiv -(a + b), \qquad § 20,$$

it follows that addition and subtraction of positive and negative numbers are reducible to these operations *as found in arithmetic*, where all numbers added and subtracted are positive, and where the subtrahend is never greater than the minuend.

E.g.

$$5 + (-8) = 5 - 8 = -(8 - 5) = -3.$$

$$5 - (-8) = 5 + 8 = 13.$$

$$-5 - 8 = -(5 + 8) = -13.$$

THEOREMS ON MULTIPLICATION AND DIVISION

22. Theorem 1. *The product of any number and zero is zero. That is,* $a \cdot 0 \equiv 0$.

Proof. By definition of zero, § 6, $a \cdot 0 = a(b - b)$.

By the distributive law of multiplication, § 10,

$$a(b - b) = ab - ab,$$

which by definition is zero.

Hence $a \cdot 0 = 0$.

Notice that by the commutative law of multiplication, § 8,

$$a \cdot 0 = 0 \cdot a.$$

It follows from this theorem and § 9, that a product is zero if any one of its factors is zero; and conversely, by § 11, if a product is zero, then at least one of its factors must be zero.

23. Corollary 1. $\frac{0}{a} = 0$, *provided* a *is not zero.*

Since by the theorem $0 = a \cdot 0$, the corollary is an immediate consequence of the definition of division (§ 11).

24. Corollary 2. $\frac{0}{0}$ *represents any number whatever. That is,* $\frac{0}{0} \equiv k$, *for all values of* k.

Since $0 \equiv 0 \cdot k$, this is an immediate consequence of the definition of division.

25. Corollary 3. *There is no number* k *such that* $\frac{a}{0} = k$, *provided* a *is not zero.*

This follows at once from $k \cdot 0 = 0$ for all values of k.

From §§ 24, 25, it follows that *division by zero is to be ruled out in all cases* unless special interpretation is given to the results thus obtained.

26. Theorem 2. $a(-b) = -ab$. See § 63, E. C.

Proof. Let $a(-b) = x$. (1)

By § 2, $a(-b) + ab = x + ab$. (2)

By § 10, $a[(-b) + b] = x + ab$. (3)

By §§ 14, 22, $a \cdot 0 = 0 = x + ab$. (4)

By § 14, $x = -ab$. (5)

Hence, from (1) and (5) $a(-b) = -ab$. (6)

27. Theorem 3. $(-a)(-b) = ab$. See § 63, E. C.

Proof. Let $(-a)(-b) = x$. (1)

By §§ 2 and 26, $(-a)(-b) + (-a)b = x - ab$. (2)

By §§ 10, 14, 22, $(-a)(-b + b) = 0 = x - ab$. (3)

Hence, § 14, $x = ab$. (4)

From (1) and (4) $(-a)(-b) = ab$. (5)

28. Theorem 4. *If the signs of the dividend and divisor are alike, the quotient is positive; and if unlike, the quotient is negative.* See § 67, E. C.

Proof. This theorem is an immediate consequence of the definition of division and the identities,

$$(+a)(+b) \equiv ab. \quad a(-b) \equiv -ab \text{ and } (-a)(-b) \equiv ab.$$

29. Theorem 5. $\frac{b}{c} \cdot a = \frac{ba}{c}$. See § 191, E. C.

Proof. Let $x = \frac{b}{c} \cdot a.$ (1)

By §§ 7 and 11, $cx = c \cdot \frac{b}{c} \cdot a = ba.$ (2)

Dividing by c, § 11, $x = \frac{ba}{c}.$ (3)

Hence from (1) and (3) $\frac{b}{c} \cdot a = \frac{ba}{c}.$ (4)

30. Theorem 6. $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$, *and* $\frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$. See § 25, E. C.

Proof. By § 29, $\frac{a+b}{c} = \frac{1 \cdot (a+b)}{c} = \frac{1}{c} \cdot (a+b).$ (1)

By §§ 10 and 29, $\frac{1}{c} \cdot (a+b) = \frac{1}{c} \cdot a + \frac{1}{c} \cdot b = \frac{a}{c} + \frac{b}{c}.$ (2)

From (1) and (2), $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}.$ (3)

Similarly we may show that $\frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}.$

This theorem states what is called the **distributive law of division.**

31. In the proofs of the above theorems certain axioms have been assumed to hold in a *more general form* than the one in which they are stated. For example, the commutative law of addition was stated for two numbers only and has been assumed for more than two. These extensions can be shown to follow from the axioms as given. It has likewise been assumed that zero and unity, which were defined respectively as $a - a = 0$, and $\frac{a}{a} = 1$, are the same for all values of a.

CHAPTER II

FUNDAMENTAL OPERATIONS

32. The operations of addition, subtraction, multiplication, division, and finding powers and roots are called **algebraic operations.**

33. An **algebraic expression** is any combination of number symbols (Arabic figures or letters or both) by means of indicated algebraic operations.

E.g. 24, $3+7$, $9(b+c)$, $\dfrac{m+n}{k}$, $x^2+\sqrt{y}$, are algebraic expressions.

34. Any number symbol upon which an algebraic operation is to be performed is called an **operand.**

All the algebraic operations have been used in the Elementary Course. They are now to be considered in connection with the fundamental laws developed in the preceding chapter, and then applied to more complicated expressions. The finding of powers and roots will be extended to higher cases.

35. One of the two equal factors of an expression is called the **square root** of the expression; one of the three equal factors is called its **cube root**; one of the four equal factors, its **fourth root,** etc. A root is indicated by the **radical sign** and a number, called the **index** of the root, which is written within the sign. In the case of the square root, the index is omitted.

E.g. $\sqrt{4}$ is read *the square root of* 4; $\sqrt[3]{8}$ is read *the cube root of* 8; $\sqrt[4]{64}$ is read *the fourth root of* 64, etc.

36. A root which can be expressed in the *form* of an integer, or as the quotient of two integers, is said to be **rational,** while one which cannot be so expressed is **irrational.**

E.g. $\sqrt[3]{8}=2$, $\sqrt{a^2+2ab+b^2}=a+b$, and $\sqrt{\frac{9}{4}}=\frac{3}{2}$ are rational roots, while $\sqrt[3]{4}$ and $\sqrt{a^2+ab+b^2}$ are irrational roots.

An algebraic expression which involves a letter in an irrational root is said to be **irrational with respect to that letter**; otherwise the expression is rational with respect to the letter.

E.g. $a + b\sqrt{c}$ is rational with respect to a and b, and irrational with respect to c.

37. An expression is **fractional** with respect to a given letter if after reducing its fractions to their lowest terms the letter is still contained in a denominator.

E.g. $\dfrac{a}{c+d} + b$ is fractional with respect to c and d, but not with respect to a and b.

38. **Order of Algebraic Operations.** In a series of indicated operations where no parentheses or other symbols of aggregation occur, it is an established usage that the operations of finding powers and roots are to be performed first, then the operations of multiplication and division, and finally the operations of addition and subtraction.

$$E.g.\ 2 + 3 \cdot 4 + 5 \cdot \sqrt[3]{8} - 4^2 \div 8 = 2 + 3 \cdot 4 + 5 \cdot 2 - 16 \div 8$$
$$= 2 + 12 + 10 - 2 = 22.$$

In cases where it is necessary to distinguish whether multiplication or division is to be performed first, parentheses are used.

E.g. In $6 \div 3 \times 2$, if the division comes first, it is written $(6 \div 3) \times 2 = 4$, and if the multiplication come first, it is written $6 \div (3 \times 2) = 1$.

ADDITION AND SUBTRACTION OF MONOMIALS

39. In accordance with § 10, the sum (or difference) of terms which are *similar with respect to a common factor* (§ 78, E. C.) is equal to the product of this common factor and the sum (or difference) of its coefficients.

Ex. 1. $8\,ax^2 + 9\,ax^2 - 3\,ax^2 = (8 + 9 - 3)\,ax^2 = 14\,ax^2.$

Ex. 2. $a\sqrt{x^2+y^2} + b\sqrt{x^2+y^2} = (a+b)\sqrt{x^2+y^2}.$

Ex. 3. $$\frac{x(x-1)(x-2)}{1 \cdot 2 \cdot 3} + \frac{x(x-1)}{1 \cdot 2} = \left(\frac{x-2}{3} + 1\right)\frac{x(x-1)}{1 \cdot 2}$$
$$= \frac{x+1}{3} \cdot \frac{x(x-1)}{1 \cdot 2} = \frac{(x+1)x(x-1)}{1 \cdot 2 \cdot 3}.$$

EXERCISES

Perform the following indicated operations:

1. $5x^4b^2 - 3x^4b^2 - 4x^4b^2 + 7x^4b^2$.

2. $3\sqrt{x^2-4} - 2\sqrt{x^2-4} + 2\sqrt{x^2-4} - 4\sqrt{x^2-4}$.

3. $ab^5c^4 - db^5c^4 + eb^5c^4 + fb^5c^4$.

4. $a^6x^4 + 5a^5x^4 - 5a^5x^5 - 3a^5x^4$.

5. $7x^3y^5 + 5x^4y^4 - 9x^4y^5 + 5x^3y^4$.

6. $2a^n + a^{n-1} + a^{n+1} = a^{n-1}(2a + 1 + a^2) = a^{n-1}(1 + a)^2$.

7. $n(n-1)(n-2)(n-3)(n-4) + n(n-1)(n-2)(n-3)$.

$n(n-1)(n-2)(n-3)$ is the common factor and $n-4$ and 1 are the coefficients to be added.

8. $n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)$
$+ n(n-1)(n-2)(n-3)(n-4)(n-5)$.

9. $n(n-1)(n-2)(n-3) + (n-1)(n-2)(n-3)$.

10. $n(n-1)(n-2)(n-3)(n-4) + (n-1)(n-2)(n-3)$.

11. $(a-4)(b+3) + (a-1)(b-2) + (a+3)(b+3)$.

First add $(a-4)(b+3)$ and $(a+3)(b+3)$.

12. $(x+2y)(x-2y) + (x-3y)(x-2y) - (2x-y)(x-y)$.

13. $(5a-3b)(a-b)(a+b) + (2b-4a)(a-b)(a+b)$
$+ (a-b)^2(2a-b)$.

14. $(7x^2+3y^2)(5x-y)(x+y) + (7x^2+3y^2)(x+y)(2y-4x)$
$+ (7x^2-3y^2)(x+y)^2$.

15. $2^3 \cdot 3^2 \cdot 5 + 2^4 \cdot 3 \cdot 5$.

The common factor is $2^3 \cdot 3 \cdot 5$. Hence the sum is

$$2^3 \cdot 3 \cdot 5(3+2) = 2^3 \cdot 3 \cdot 5^2.$$

16. $2 \cdot 3^4 \cdot 7 + 2^2 \cdot 3^3 \cdot 7^2 - 2^4 \cdot 3^3 \cdot 7$.

17. $3^4 \cdot 5^7 \cdot 13 + 3^5 \cdot 5^7 \cdot 13^2$.

18. $5^4 \cdot 7^3 \cdot 11 + 5^3 \cdot 7^2 \cdot 11 - 2^3 \cdot 3 \cdot 5^3 \cdot 7^2 \cdot 11.$

19. $3^{22} \cdot 7^{18} \cdot 13^{15} + 3^{21} \cdot 7^{17} \cdot 13^{15} + 3^{24} \cdot 7^{17} \cdot 13^{15}.$

20. $1 \cdot 2 \cdot 3 \cdots n \ + 1 \cdot 2 \cdot 3 \cdots n(n+1).$

The dots mean that the factors are to run on in the manner indicated up to the number n. The common factor in this case is $1 \cdot 2 \cdot 3 \cdots n$, and the coefficients to be added are 1 and $n+1$. Hence the sum is $1 \cdot 2 \cdot 3 \cdots n(n+2)$.

21. $1 \cdot 2 \cdot 3 \cdots n \ + 1 \cdot 2 \cdot 3 \cdots n(n+1)$
$\qquad + 1 \cdot 2 \cdot 3 \cdots n(n+1)(n+2).$

22. $1 \cdot 2 \cdot 3 \cdots n \ + 3 \cdot 4 \cdot 5 \cdots n \ + 5 \cdot 6 \cdot 7 \cdots n.$

23. $n(n-1) \cdots (n-6) \ + n(n-1) \cdots (n-6)(n-7).$

24. $n(n-1) \cdots (n-r) \ + n(n-1) \cdots (n-r)(n-r-1).$

25. $na^nb + a^nb.$

26. $\dfrac{n(n-1)}{1 \cdot 2} a^{n-1}b^2 + na^{n-1}b^2.$

27. $\dfrac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-2}b^3 + \dfrac{n(n-1)}{1 \cdot 2} a^{n-2}b^3.$

The common factor is $\dfrac{n(n-1)}{1 \cdot 2} a^{n-2}b^3$ and the coefficients to be added are $\dfrac{n-2}{3}$ and 1.

28. $\dfrac{n(n-1)(n-2)(n-3)}{2 \cdot 3 \cdot 4} a^{n-3}b^4 + \dfrac{n(n-1)(n-2)}{2 \cdot 3} a^{n-3}b^4.$

29. $\dfrac{n(n-1)(n-2)(n-3)(n-4)}{2 \cdot 3 \cdot 4 \cdot 5} a^{n-4}b^5$
$\qquad + \dfrac{n(n-1)(n-2)(n-3)}{2 \cdot 3 \cdot 4} a^{n-4}b^5.$

30. $\dfrac{n(n-1) \cdots (n-r+1)(n-r)}{2 \cdot 3 \cdots r(r+1)} a^{n-r}b^{r+1}$
$\qquad + \dfrac{n(n-1) \cdots (n-r+1)}{2 \cdot 3 \cdots r} a^{n-r}b^{r+1}.$

ADDITION AND SUBTRACTION OF POLYNOMIALS

40. The **addition** of polynomials is illustrated by the following example.

Add $2a+3b-4c$ and $3a-2b+5c$.

The sum may be written thus:

$$(2a+3b-4c)+(3a-2b+5c).$$

By the associative law, § 5, and by § 16, we have,

$$2a+3b-4c+3a-2b+5c.$$

By the commutative law, § 4, and by § 15, this becomes,

$$2a+3a+3b-2b-4c+5c.$$

Again by the associative law, combining similar terms, we have,

$$5a+b+c.$$

From this example it is evident that several polynomials may be added by combining similar terms and then indicating the sum of these results.

For this purpose the polynomials are conveniently arranged so that similar terms shall be in the same column. Thus, in the above example,

$$\begin{array}{r} 2a+3b-4c \\ 3a-2b+5c \\ \hline 5a+\ \ b+\ \ c \end{array}$$

41. For **subtraction** the terms of the polynomials are arranged as for addition. The subtraction itself is then performed as in the case of monomials. See §§ 17–19.

EXAMPLE. Subtract $4x-2y+6z$ from $3x+6y-3z$.

$$\begin{array}{r} 3x+6y-3z \\ 4x-2y+6z \\ \hline -x+8y-9z \end{array}$$

The steps are:

$$3x-4x=-x;\ \ 6y-(-2y)=8y;\ \ -3z-(+6z)=-9z.$$

EXERCISES

1. Add $8x^3 - 11x - 7x^2$, $2x - 6x^2 + 10$, $-5 + 4x^3 + 9x$, and $13x^2 - 5 - 12x^3$.

2. Add $5a^3 - 2a - 12 - 10a^2$, $14 - 7a + a^2 - 9a^3$, $3a^2 - 13a^3 + 4 - 11a$, and $3 - 7a + 10a^2 + 4a^3$.

3. From the sum of $9m^3 - 3m^2 + 4m - 7$ and $3m^2 - 4m^3 + 2m + 8$ subtract $4m^3 - 2m^2 - 4 + 8m$.

4. From the sum of $x^4 - ax^3 - a^2x^2 - a^3x + 2a^4$ and $3ax^3 + 7a^2x^2 - 5a^3x + 2a^4$ subtract $3x^4 + ax^3 - 3a^2x^2 + a^3x - a^4$.

5. Add $37a - 4b - 17c + 15d - 6f - 8h$ and $3c - 31a + 9b - 5d - h - 4f$.

6. Add $11q - 10p - 8n + 3m$, $24m - 17q + 15p - 13n$, $9n - 6m - 4q - 7p - 5n$, and $8q - 4p - 12m + 18n$.

7. From the sum of $13a - 15b - 7c - 11d$ and $7a - 6b + 8c + 3d$ subtract the sum of $6d - 5b - 7c + 2a$ and $5c - 10d - 28b + 17a$.

8. Add $2^3 \cdot 3^4 x^3 - 2^5 \cdot 3^2 x^2 + 2^2 \cdot 3^3 \cdot 7x + 2^2 \cdot 3^2 \cdot 5$, $2^2 \cdot 3^3 x^3 - 2^4 \cdot 3^2 \cdot 7x + 2^4 \cdot 3^3 x^2 - 2^2 \cdot 3^2 \cdot 5^2$, and $2^3 \cdot 3^3 x^3 - 2^3 \cdot 3^3 \cdot 5 + 2^3 \cdot 3^3 x - 2^4 \cdot 3^4 x^2$.

9. Add $(a + b - c)m + (a - b + c)n + (a - b - c)k$,

$(2a - 3b + c)m + (b - 3a + c)n + (4c + 2b + a)k$,

and $(b - 2c)m + (2a - 2c + b)n + (2b - 2a + c)k$.

10. From the sum of $ax^3 - bx^2 + cx - d$ and $bx^3 + ax^2 - dx + c$ subtract $(a - b)x^3 + (c - a)x^2 - (b + d)x - d + c$.

11. From $(m - n)(m - n)x^3 + (n - m)^2x^2 - (n + m)x + 8$ subtract the sum of $n(m - r)x^3 - 4(n - m)^2x^2 + (n + m)x - 31$ and $2(n - m)^2x^2 - m(m - n)x^3 - 2(n + m)x + 25$.

12. Add $a^n + 2a^{n+1} + a^{n+2}$ and $2a^n - 4a^{n+1} + 5a^{n+2}$ and from this sum subtract $7a^{n+1} - 8a^n + a^{n+2}$.

REMOVAL OF PARENTHESES

42. By the theorems of §§ 15–18, a parenthesis inclosing a polynomial may be removed with or without the change of sign of each term included, according as the sign $-$ or $+$ precedes the parenthesis.

In case an expression contains signs of aggregation, one within another, these may be removed *one at a time*, beginning with the *innermost*, as in the following example:

$$\begin{aligned} &a - \{b + c - [d - e + f - (g - h)]\} \\ &= a - \{b + c - [d - e + f - g + h]\} \\ &= a - \{b + c - d + e - f + g - h\} \\ &= a - b - c + d - e + f - g + h. \end{aligned}$$

Such involved signs of aggregation may also be removed *all at once*, beginning with *the outermost*, by observing the *number of minus signs* which affect each term, and calling the sign of any term $+$ if this number is *even*, $-$ if this number is *odd*.

Thus, in the above example, b and c are each affected by *one* minus sign, namely, the one preceding the brace. Hence we write, $a - b - c$.

d and f are each affected by *two* minus signs, namely the one before the brace and the one before the bracket, while e is affected by these two, and also by the one preceding it. Hence we write, $d - e + f$.

g is affected by the minus signs before the bracket, the brace, and the parenthesis, an *odd* number, while h is affected by these and also by the one preceding it, an *even* number. Hence we write $-g + h$.

By counting in this manner as we proceed from left to right, we give the final form at once, $a - b - c + d - e + f - g + h$.

EXERCISES

In removing the signs of aggregation in the following, either process just explained may be used. The second method is shorter and should be easily followed after a little practice.

1. $7 - \{-4 - (4 - [-7]) - (5 - [4 - 5] + 2)\}$.

2. $-[-(7-\{-4+9\}-13)-(12-3+[-7+2])]$.

3. $6-(-3-[-5+4]+\{7-3-(7-19)\}+8)$.

4. $5+[-(-\{-5-3+11\}-15)-3]+8$.

5. $4x-[3x-y-\{3x-y-(x-\overline{y-x})+x\}-3y]$.

The vinculum above $y-x$ has the same effect as a parenthesis, *i.e.* $-\overline{y-x}=-(y-x)$.

6. $3x^2-2y^2-(4x^2-\{3x^2-(y^2-2x^2)-3y^2\}-y^2+4x^2)$.

7. $7a-\{3a-[-2a-\overline{a+3}+a]-\overline{2a-5}\}$.

8. $l-(-2m-n-\{l-m\})-(5l-2n-[-3m+n])$.

9. $2d-[3d+\{2d-(e-5d)\}-(d+3e)]$.

10. $4y-(-2y-[-3y-\{-y-\overline{y-1}\}+2y])$.

11. $3x-[8x-(x-3)-\{-2x+6-8\overline{x-1}\}]$.

12. $x-(x-\{-4x-[5x-\overline{2x-5}]-[-x-\overline{x-3}]\})$.

13. $3x-\{y-[3y+2z]-(4x-[2y-3z]-\overline{3y-2z})+4x\}$.

14. $x-(-x-\{-3x-[x-\overline{2x+5}]-4\}-[2x-\overline{x-3}])$.

MULTIPLICATION OF MONOMIALS

43. Theorem. *The product of two powers of the same base is a power of that base whose exponent is the sum of the exponents of the common base.* See § 127, E. C.

Proof. Let b be any number and k and n any positive integers. It is to be proved that

$$b^k \cdot b^n = b^{k+n}.$$

By the definition of a positive integral exponent,

$$b^k = b \cdot b \cdot b \cdots \text{ to } k \text{ factors},$$

and

$$b^n = b \cdot b \cdot b \cdots \text{ to } n \text{ factors}.$$

Hence,

$$b^k \cdot b^n = (b \cdot b \cdots \text{ to } k \text{ factors})(b \cdot b \cdots \text{ to } n \text{ factors})$$
$$= b \cdot b \cdot b \cdots \text{ to } k+n \text{ factors},$$

since the factors may be associated in a single group, § 9.

Hence, by the definition of a positive integral exponent, we have,

$$b^k \cdot b^n = b^{k+n}.$$

44. In finding the product of two monomials, the factors may be *arranged* and *associated* in any manner, according to §§ 8, 9.

E.g. $(3ab^2) \times (5a^2b^3) = 3ab^2 \cdot 5a^2b^3$ § 9

$= 3 \cdot 5 \cdot a \cdot a^2 \cdot b^2 \cdot b^3$ § 8

$= (3 \cdot 5)(a \cdot a^2)(b^2 \cdot b^3)$ § 9

$= 15a^3b^5$ by the theorem, § 43

The factors in the product are arranged so as to associate those consisting of Arabic figures and also those which are powers of the same base. This arrangement and association of the factors is equivalent to multiplying either monomial by the factors of the other in succession. See § 129, E. C.

45. It is readily seen that a product is negative when it contains an *odd* number of *negative* factors; otherwise it is positive.

For by the commutative and associative laws of factors the negative factors may be grouped in *pairs*, each pair giving a *positive* product. If the number of negative factors is odd, there will be just one remaining, which makes the final product negative.

EXERCISES

Find the products of the following:

1. $2^3 \cdot 3^4 \cdot 4^7,\ 2^7 \cdot 3^2 \cdot 4^2$.
2. $3 \cdot 2^4 \cdot 5^2,\ 5 \cdot 2^2 \cdot 5,\ 7 \cdot 2^3 \cdot 5^3$.
3. $2x^2y^3,\ 5x^3y^2,\ 2x^4y$.
4. $5xy,\ 2x^3y,\ 4xy^5,\ x^2y^2$.
5. $3a^5bc,\ ab^2c,\ a^2bc^4,\ 4ab^5c$.
6. $x^n,\ x^{n-1},\ x^{n+1},\ 2x^n$.
7. $x^{m+n-1},\ x^{m-n+1},\ x^{2m}$.
8. $a^x,\ a^{3x-y},\ a^{y-3x}$.
9. $a^nb^m,\ a^{2n}b^{3m},\ a^{1-3n}b^{2-4m}$.
10. $4ab^m,\ 2a^3b^n,\ 3a^6b^{2-m-n}$.
11. $2x^my^{m+n},\ 3x^{m-1}y^{2n-m+2}$.
12. $a^{d-2c+2}b^{m-3n},\ a^{2c-d-1}b^{2-m+3n}$.
13. $3x^{a+3b},\ 2x^{a-2b}y^{c-3},\ 2x^{4-2a-b}y^{2c+3}$.
14. $a^{2x-3}b^{y+1},\ a^{x+3}b^{y-1},\ 3a^3b^2$.
15. $3^{4a-2-2b} \cdot 2^{n+3-m},\ 3^{5-4a+2b} \cdot 2^{m+2-n}$.
16. $x^{3x+1+y},\ x^{y-2x-1}y^{2x},\ y^{1-x}$.
17. $7 \cdot 2^{3a-4},\ 3 \cdot 2^{5-2a},\ 5 \cdot 2^{3-a}$.
18. $2^{7x-1-4y},\ 3 \cdot 2^{1-5x-4y},\ 3^2 \cdot 2^{2-2x}$.
19. $3^{2-5m+3n} \cdot 2^{4a-3b},\ 3^{2-3n+6m} \cdot 2^{5+3b+5a}$.
20. $(1+a)^{7-3b+a} \cdot (1-a)^{2+a-b},\ (1-a)^{b-a-1} \cdot (1+a)^{3b-a-6}$.

DIVISION OF MONOMIALS

46. Theorem 1. *The quotient of two powers of the same base is a power of that base whose exponent is the exponent of the dividend minus that of the divisor.* See § 154, E. C.

Proof. Let a be any number and let m and k be positive integers of which m is the greater. We are to prove,

$$a^m \div a^k = a^{m-k}.$$

Since k and $m-k$ are both positive integers, we have, by § 43, $a^k a^{m-k} = a^{k+m-k} = a^m$. That is, a^{m-k} is the number which multiplied by a^k gives a product a^m, and hence by the definition of division,

$$a^m \div a^k = a^{m-k}.$$

Under the proper interpretation of negative numbers used as exponents this theorem also holds when $m < k$. This is considered in detail in § 177. We remark here that in case $m = k$, the dividend and the divisor are equal and the quotient is unity. Hence $a^m \div a^m = a^{m-m} = a^0 = 1$. See § 11.

47. Theorem 2. *In dividing one algebraic expression by another, all factors common to dividend and divisor may be removed or canceled.* See §§ 23, 156, 157, E. C.

Proof. We are to show that $\dfrac{ak}{bk} = \dfrac{a}{b}$.

By definition of division, § 11, $\dfrac{ak}{bk} \cdot bk = ak.$ (1)

Also $\dfrac{a}{b} \cdot b = a.$ (2)

Multiplying (2) by k, $\dfrac{a}{b} \cdot bk = ak.$ (3)

Hence from (1) and (3), $\dfrac{ak}{bk} \cdot bk = \dfrac{a}{b} \cdot bk.$ (4)

Dividing by bk, $\dfrac{ak}{bk} = \dfrac{a}{b}.$ (5)

EXERCISES

Divide:

1. $4 \cdot 2^4 \cdot 3^7 \cdot 5^2$ by $3 \cdot 2^3 \cdot 3^4 \cdot 5$.

2. $5 \cdot 3^7 \cdot 7^4 \cdot 13^5$ by $2 \cdot 3^5 \cdot 7^2 \cdot 13^2$.

3. $3x^7y^2z$ by $2x^3yz$.

4. $5a^5b^7c^8$ by $5a^4b^7c^4d^2$.

5. $x^{2n}y^mz^{3m}$ by $x^ny^mz^m$.

6. $a^{3n-5}y^{2n+3}$ by $a^{n+6}y^{2n+1}$.

7. $a^{c+3d+2}b^{d-2c+6}$ by a^{c+2d-4}.

8. $3^{a+2b-7} \cdot 5^{3b-2a+4}$ by $3^{b+a-8} \cdot 5^{2b-2a+3}$.

9. $a^{3+2m-3n}b^5c^{7-n}$ by $a^{2+m-4n}b^4c^{7-n}$.

10. $x^{4a-2b+1}y^{c-a+b}z^{3a+2b+c}$ by $z^{2b-c+3a}y^{a-c+b}x^{3a+b-2c}$.

11. $2^{3a-4+7b} \cdot 3^{3b-4c+6}$ by $2^{2a-5-7b} \cdot 3^{2b-6c+7}$.

12. $(x-2)^{3m+1-3n} \cdot (x+2)^{2m+2-3n}$ by $(x+2)^{1+2m-2n} \cdot (x-2)^{1-3n+2m}$.

13. $(x-y)^{5b-3c-1} \cdot (x+y)^{7c-2b+2}$ by $(x-y)^{-2-3c+5b} \cdot (x+y)^{-3-2b+7c}$.

14. $(a^2-b^2)^{3+4k+7b} \cdot (a^2-b^2)^{1-3k-5b}$ by $(a^2-b^2)^{4+k} \cdot (a^2-b^2)^{-2+2b}$.

MULTIPLICATION OF POLYNOMIALS

48. Theorem. *The product of two polynomials is equal to the sum of the products obtained by multiplying each term of one polynomial by every term of the other.* See § 86, E. C.

Proof. By the distributive law, § 10, we have,

$$\begin{aligned}(m+n+k)(a+b+c) &= m(a+b+c)\\ &\quad + n(a+b+c)\\ &\quad + k(a+b+c).\end{aligned}$$

Applying the same law to each part, we have the product,

$$ma + mb + mc + na + nb + nc + ka + kb + kc.$$

This is Principle XIII of the Elementary Course.

EXERCISES

Find the following indicated products:

1. $(a+b)(a+b)$, i.e. $(a+b)^2$; also $(a-b)^2$.
2. $(a+b)(a+b)(a+b)$, i.e. $(a+b)^3$; also $(a-b)^3$.
3. $(a+b)(a+b)(a+b)(a+b)$, i.e. $(a+b)^4$; also $(a-b)^4$.
4. $(a^2+2ab+b^2)(a^2+2ab+b^2)(a+b)$.
5. $(a^2-2ab+b^2)(a^2-2ab+b^2)(a-b)$.
6. $(a^2+2ab+b^2)^3$; also $(a+b)^6$.
7. $(a^3+3a^2b+3ab^2+b^3)^2$.
8. $(a^3-3a^2b+3ab^2-b^3)^2$.
9. $(a-b)(a^2+ab+b^2)$.
10. $(a+b)(a^2-ab+b^2)$.
11. $(a^2+2ab+b^2)(a^4+4a^3b+6a^2b^2+4ab^3+b^4)$.
12. $(a^2-2ab+b^2)(a^4-4a^3b+6a^2b^2-4ab^3+b^4)$.
13. $(a-b)(a^3+a^2b+ab^2+b^3)$.
14. $(a+b)(a^3-a^2b+ab^2-b^3)$.
15. $(a-b)(a^4+a^3b+a^2b^2+ab^3+b^4)$.
16. $(a+b)(a^4-a^3b+a^2b^2-ab^3+b^4)$.
17. $(a-b)(a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5)$.
18. $(a+b)(a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5)$.
19. $(1-r)(a+ar+ar^2+ar^3)$.
20. $(1-r)(a+ar+ar^2+ar^3+ar^4+ar^5)$.
21. $(a+b+c)^2$.
22. $(a+b-c)^2$.
23. $(a-b-c)^2$.
24. From Exs. 21–23 deduce a rule for squaring a trinomial.
25. $(x+y+z+v)^2$.
26. $(x-y+z-v)^2$.
27. From Exs. 25, 26 deduce a rule for squaring a polynomial.
28. $(a+b+c)(a+b-c)(a-b+c)(b-a+c)$.
29. $(ab+ac+bc)(ab+ac-bc)(ab-ac+bc)(ac+bc-ab)$.
30. $(a-b+c+d)(a+b+c-d)(a+b-c+d)(-a+b+c+d)$.

31. $(4x^2-6xy+9y^2)(2x+3y)(4x^2+6xy+9y^2)(2x-3y)$.

32. Collect in a table the following products:

$(a+b)^2$, $(a-b)^2$, $(a+b)^3$, $(a-b)^3$, $(a+b)^4$.
$(a-b)^4$, $(a+b)^5$, $(a-b)^5$, $(a+b)^6$, $(a-b)^6$.

33. From the above table answer the following questions:

(*a*) How many terms in each product, compared with the exponent of the binomial?

(*b*) Tell how the signs occur in the various cases.

(*c*) How do the exponents of a proceed? of b?

(*d*) Make a table of the coefficients alone and memorize this.

E.g. For $(a+b)^5$, they are 1, 5, 10, 10, 5, 1.

34. Make use of the rules in Exs. 24, 27, 33 to write the following products: (*a*) $(2x-3y+4z)^2$, (*b*) $(\frac{1}{2}m^2-\frac{1}{4}n^3-3r)^2$, (*c*) $(4ax-2ay+3m-n)^2$.

(*d*) $(2x+3y)^3$.

(*e*) $(3a-b^2)^4$.

(*f*) $(\frac{1}{2}ax-\frac{2}{3}by)^3$.

(*g*) $(2m-n)^5$.

(*h*) $\left(\frac{a}{3}-\frac{b}{4}\right)^4$.

(*i*) $\left(2x-\frac{y}{2}\right)^5$.

(*j*) $\left(\frac{a}{2}-x^2\right)^3$.

(*k*) $(9x-2y)^2$.

(*l*) $(1-2x)^4$.

(*m*) $(1+3x)^5$.

DIVISION OF POLYNOMIALS

49. According to the distributive law of division, § 30, a *polynomial is divided by a monomial* by dividing each term separately by the monomial. See § 25, E. C.

$$E.g.\quad \frac{ab+ac-ad}{a}=\frac{ab}{a}+\frac{ac}{a}-\frac{ad}{a}=b+c-d.$$

A polynomial is divided by a polynomial by separating the dividend into polynomials, each of which is the product of the divisor and a monomial. Each of these monomial factors is a part of the quotient, their sum constituting the whole quotient. The parts of the dividend are found one by one as the work proceeds. See §§ 161–163, E.C. This is best shown by an example.

$$
\begin{array}{lrl}
\text{Dividend,} & a^4 + a^3 - 4a^2 + 5a - 3 & \underline{|a^2 + 2a - 3,\ \text{Divisor.}} \\
\text{1st part of dividend:} & \underline{a^4 + 2a^3 - 3a^2} & |a^2 - a + 1, \quad \text{Quotient.} \\
 & -a^3 - a^2 + 5a - 3 & \\
\text{2d part of dividend:} & \underline{-a^3 - 2a^2 + 3a} & \\
 & a^2 + 2a - 3 & \\
\text{3d part of dividend:} & \underline{a^2 + 2a - 3} & \\
 & 0 &
\end{array}
$$

The three parts of the dividend are the products of the divisor and the three terms of the quotient. If after the successive subtraction of these parts of the dividend the remainder is zero, the division is exact. In case the division is not exact, there is a final remainder such that

$$\textit{Dividend} = \textit{Quotient} \times \textit{divisor} + \textit{Remainder}.$$

In symbols we have $\mathbf{D} = \mathbf{Q} \cdot \mathbf{d} + \mathbf{R}$.

EXERCISES

Divide:

1. $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$ by $x^2 + 2xy + y^2$.
2. $x^8 + x^4y^4 + y^8$ by $x^4 - x^2y^2 + y^4$.
3. $x^5 - y^5$ by $x - y$.
4. $x^8 - y^8$ by $x^3 + x^2y + xy^2 + y^3$.
5. $x^9 + y^9$ by $x^2 - xy + y^2$.
6. $x^8 - y^8$ by $x^2 - y^2$.
7. $a^6 + b^6$ by $a^2 + b^2$.
8. $x^{6a} - y^{6b}$ by $x^{2a} - y^{2b}$.
9. $a^{10} - a^5b^5 + b^{10}$ by $a^2 - ab + b^2$.
10. $2x^4 - 3x^3b + 6x^2b^2 - xb^3 + 6b^4$ by $x^2 - 2xb + 3b^2$.
11. $2x^6 - 5x^5 + 6x^4 - 6x^3 + 6x^2 - 4x + 1$ by $x^4 - x^3 + x^2 - x + 1$.
12. $26a^3b^3 + a^6 + 6b^6 - 5a^5b - 17ab^5 - 2a^4b^2 - a^2b^4$ by $a^2 - 3b^2 - 2ab$.
13. $x^4 + 2x^3 - 7x^2 - 8x + 12$ by $x^2 - 3x + 2$.
14. $4b^2 + 4ab + a^2 - 12bc - 6ac + 9c^2$ by $2b + a - 3c$.
15. $x^4 + 4xy^3 - 4xyz + 3y^4 + 2y^2z - z^2$ by $x^2 - 2xy + 3y^2 - z$.
16. $a^2b^2c + 3a^2b^3 - 3abc^3 - a^2c^3 + b^5 - 4b^3c^2 + 3ab^3c + 3bc^4 - 3a^2bc^2$ by $b^2 - c^2$.

CHAPTER III

INTEGRAL EQUATIONS OF THE FIRST DEGREE IN ONE UNKNOWN

50. When in an algebraic expression a letter is replaced by another number symbol, this is called a **substitution on that letter.**

E.g. In the expression, $2a + 5$, if a is replaced by 3, giving $2 \cdot 3 + 5$, this is a substitution on the letter a.

51. An equality containing a single letter is said to be satisfied by any substitution on that letter which reduces both members of the equality to the same number.

E.g. $4x + 8 = 24$ is satisfied by $x = 4$, since $4 \cdot 4 + 8 = 24$.

We notice, however, that the substitution *must not reduce the denominator of any fraction to zero.*

Thus $x = 2$ does *not* satisfy $\dfrac{x^2 - 4}{x - 2} = 8$ although it reduces the left member of the equation to $\dfrac{0}{0}$, which by § 24 equals 8 or *any other number whatever.*

On the other hand, $x = 6$ satisfies this equation, since

$$\frac{6^2 - 4}{6 - 2} = \frac{32}{4} = 8.$$

52. An equality in two or more letters is **satisfied** by any simultaneous substitutions on these letters which reduce both members to the same number.

E.g. $6a + 3b = 15$ is satisfied by $a = 2, b = 1$; $a = \frac{3}{2}, b = 2$; $a = 1, b = 3$, etc.

$\dfrac{x^2 - y^2}{x^2 + 2xy + y^2} = \dfrac{1}{2}$ is satisfied by $x = 3, y = 1$, but is *not* satisfied by any values of x and y such that $x = -y$, since these reduce the denominator (and also the numerator) to zero. See § 24.

53. An equality is said to be an **identity in all its letters**, or simply an **identity**, if it is satisfied by *every possible substitution* on these letters, not counting those which make any denominator zero.

If an equality is an identity, both members will be reduced to the same expression when all indicated operations are performed as far as possible.

The members of an identity are called **identical expressions**.

Thus in the identity $(a + b)^2 \equiv a^2 + 2\,ab + b^2$, performing the indicated operation in the first member reduces it to the same form as the second.

54. An equality which is not an identity is called an **equation of condition** or simply an **equation**.

The members of an equation *cannot* be reduced to the same expression by performing the indicated operations.

E.g. $(x - 2)(x - 3) = 0$ cannot be so reduced. This is an equation which is satisfied by $x = 2$ and $x = 3$. See § 22.

55. In an *equation* containing several letters any one or more of them may be regarded as **unknown**, the remaining ones being considered **known**. Such an equation is said to be satisfied by any substitution on the *unknown* letters which reduces it to an *identity in the remaining letters*.

E.g. $x^2 - t^2 = sx + st$ is an equation in s, x, or t, or in any pair of these letters, or in all three of them.

As an equation in x it is satisfied by $x = s + t$, since this substitution reduces it to the identity in s and t,

$$s^2 + 2\,st \equiv s^2 + 2\,st.$$

As an equation in s it is satisfied by $s = x - t$, since this substitution reduces it to the identity in x and t,

$$x^2 - t^2 \equiv x^2 - t^2.$$

Any number expression which satisfies an equation in one unknown is called a **root of the equation**.

E.g. $s + t$ is a root of the equation $x^2 - t^2 = sx + st$, when x is the unknown, and $x - t$ is a root when s is the unknown.

56. An equation is **rational in a given letter** if every term in the equation is rational with respect to that letter.

An equation is **integral** in a given letter if every term is rational and integral in that letter.

57. The **degree** of a rational, integral equation in a given letter is the highest exponent of that letter in the equation.

In determining the degree of an equation according to this definition it is necessary that all indicated multiplications be performed as far as possible.

E.g. $(x-2)(x-3)=0$ is of the 2d degree in x, since it reduces to $x^2-5x+6=0$.

EQUIVALENT EQUATIONS

58. Two equations are said to be **equivalent** if every root of either is also a root of the other.

59. Theorem 1. *If one rational, integral equation is derived from another by performing the indicated operations, then the two equations are equivalent.* See § 36, E. C.

Proof. In performing the indicated operations, each expression is replaced by another identically equal to it. Hence any expression which satisfies the given equation must satisfy the other and conversely.

E.g. $10x=50$ is equivalent to $3x+7x=50$, since $3x+7x\equiv 10x$; and $8(2x-3y)=2y-1$ is equivalent to $16x-24y=2y-1$, since $8(2x-3y)\equiv 16x-24y$.

60. Theorem 2. *If any equation is derived from another by adding the same expression to each member, or by subtracting the same expression from each member, then the equations are equivalent.* See § 36, E. C.

Proof. For simplicity of statement we prove the theorem for the case where the original equation contains only one unknown, the proof in the other cases being similar.

Let
$$M=N \tag{1}$$
be an equation involving one unknown, x, and let A be an expression which may or may not involve x.

We are to show that equation (1) is equivalent to

$$M + A = N + A, \tag{2}$$

and also to

$$M - A = N - A. \tag{3}$$

(*a*) Let x_1 be a root of (1). Then substituting x_1 for x in (1) makes M and N identical. Since $A = A$ for any value of x it follows, § 2, that the substitution of x_1 reduces $M + A$ and $N + A$ to identical expressions. That is, $x = x_1$ satisfies equation (2). Hence any root of (1) is also a root of (2).

(*b*) Again if x_1 is a root of (2), its substitution reduces $M + A$ and $N + A$ to identical expressions, and hence by § 6, it also reduces $M + A - A$ and $N + A - A$ to identical expressions. That is, $x = x_1$ satisfies equation (1). Hence any root of (2) is also a root of (1). From (*a*) and (*b*) it follows that equations (1) and (2) are equivalent.

In like manner (1) and (3) are shown to be equivalent.

61. Corollary. *Any equation can be reduced to an equivalent equation of the form* $R = 0$.

For if an equation is in the form $M = N$, then by theorem **1** it is equivalent to $M - N = N - N = 0$, which is in the form $R = 0$.

62. Theorem 3. *If one equation is derived from another by multiplying or dividing each member by the same expression, then the equations are equivalent, provided the original equation is not multiplied or divided by zero or by an expression containing the unknown of the equation.*

See § 36, E. C., and the note following it.

Proof. Again consider the case where the original equation contains only one unknown.

Let A be an expression not containing x, and different from zero. We are to show that

$$M = N \tag{1}$$

is equivalent to

$$M \cdot A = N \cdot A, \tag{2}$$

and also to

$$\frac{M}{A} = \frac{N}{A}. \tag{3}$$

If x_1 is a root of (1), its substitution makes M and N identical, and hence also $M \cdot A$ and $N \cdot A$ by § 7. That is, x_1 is a root of (2). Similarly, if x_1 is a root of (2), then, by § 11, it is a root of (1).

Hence (1) and (2) are equivalent. In like manner, we may show (1) and (3) equivalent.

63. The ordinary processes of solving equations depend upon theorems **1**, **2**, and **3**, as is illustrated by the following examples:

Ex. 1. $$(x+4)(x+5)=(x+2)(x+6). \quad (1)$$

$$x^2+9x+20=x^2+8x+12. \quad (2)$$

$$x=-8. \quad (3)$$

By theorem **1**, (1) and (2) are equivalent, and by theorem **2**, (2) and (3) are equivalent. Hence (1) and (3) are equivalent. That is, -8 is the root of (1).

Ex. 2. $$\tfrac{2}{3}x+\tfrac{4}{3}=4. \quad (1)$$

$$2x+4=12. \quad (2)$$

$$2x=8. \quad (3)$$

$$x=4. \quad (4)$$

By theorem **3**, (1) and (2) are equivalent. By theorem **2**, (2) and (3) are equivalent. By theorem **3**, (3) and (4) are equivalent. Hence (1) and (4) are equivalent and 4 is the solution of (1).

These theorems are stated for equations, but they apply equally well to identities, inasmuch as the identities are changed into other identities by these operations.

64. If an *identity* is reduced to the form $R=0$, § 61, and all the indicated operations are performed, then it becomes $0=0$. See § 53. Conversely, if an equality may be reduced to the form $0=0$, it is an identity. This, therefore, is a *test* as to whether an equality is an identity.

E.g. $(x+4)^2=x^2+8x+16$ is an identity, since in $x^2+8x+16-x^2-8x-16=0$ all terms cancel, leaving $0=0$.

EXERCISES

In the following, determine which numbers or sets of numbers, if any, of those written to the right, satisfy the corresponding equation.

Remember that no substitution is legitimate which reduces any denominator to zero.

1. $4(x-1)(x-2)(x-3)=3(x-2)(x-3)$. $\quad 1, 2, 3, 4.$

2. $\dfrac{x^2-16}{x+5}=(x-4)(x+6)$. $\quad 2, 4, 6.$

3. $\dfrac{x+3}{\sqrt{x^2+7}}=x-\dfrac{3}{2}$. $\quad 2, 3, \frac{1}{2}.$

4. $\dfrac{(x-3)(x-2)}{x^2-7x+10}=x^2-5x+6$. $\quad 2, 3, 0, -2.$

5. $\dfrac{a^2+9a+20}{a^2+8a+16}=(a+4)(a-4)(a+5)$. $\quad 4, -4, 5, -5.$

6. $3a+4b=12$. $\quad \begin{cases} a=0, \\ b=3. \end{cases} \begin{cases} a=4, \\ b=0. \end{cases} \begin{cases} a=2, \\ b=2. \end{cases}$

7. $\dfrac{369(a-b)}{a^2+b^2}=a+b$. $\quad \begin{cases} a=0, \\ b=0. \end{cases} \begin{cases} a=1, \\ b=1. \end{cases} \begin{cases} a=5, \\ b=4. \end{cases}$

8. $\dfrac{x^2-y^2}{x-y}=(x-2)(y-1)$. $\quad \begin{cases} x=1, \\ y=0. \end{cases} \begin{cases} x=1, \\ y=1. \end{cases} \begin{cases} x=2, \\ y=2. \end{cases}$

9. $\dfrac{u^3-v^3}{u^2-v^2}=(u^2+uv+v^2)(u-v)$. $\begin{cases} u=1, \\ v-1. \end{cases} \begin{cases} u=1, \\ v=0. \end{cases} \begin{cases} u=-1, \\ v=0. \end{cases}$

10. $\dfrac{(r-s)(r+s)(r^2+s^2)}{r^3+rs^2-r^2s-s^3}=(r^2-s^2)(2r-3s)$. $\quad r=1, s=1;$
$r=1, s=-1; \quad r=2, s=-2; \quad r=a, s=-a.$

11. $a+b+c=6$. $\quad a=1, b=2, c=3;$
$a=3, b=3, c=0; \quad a=10, b=0, c=-4.$

12. $\dfrac{a-b+c}{\sqrt{a^2+b^2+c^2}}=\dfrac{3a-2c+5b+2}{10}$. $\quad a=8, b=0, c=6;$
$a=1, b=4, c=2; \quad a=0, b=0, c=-4.$

13. $\dfrac{(a-b)(b-c)(c-a)}{ac-bc-a^2+ba}=(b-b)(c-a)$. $\quad a=2, b=1, c=1;$
$a=3, b=2, c=3; \quad a=6, b=6, c=0.$

14. $(x-z)(x-y)(y-z)=8\,xyz(x^2-y^2)(y^2-z^2)(z^2-x^2)$.
$x=1,\ y=1,\ z=1;\quad x=1,\ y=0,\ z=1;\quad x=1,\ y=2,\ z=3.$

15. $x^3+3\,x^2y+3\,xy^2+y^3=(x+y)^3.\quad \begin{cases}x=1,\\ y=1.\end{cases}\ \begin{cases}x=1,\\ y=2.\end{cases}\ \begin{cases}x=3,\\ y=4.\end{cases}$

16. Show by reducing the equality in Ex. 15 to the form $R=O$ that it is satisfied by any pair of values whatsoever for x and y, *e.g.*, for $x=348764$, $y=594021$. What kind of an equality is this?

Which of the following four equalities are identities?

17. $12(x+y)^2+17(x+y)-7=(3\,x+3\,y-1)(4\,x+4\,y+7)$.

18. $\dfrac{a^5-b^5}{a-b}=a^4+a^3b+a^2b^2+ab^3+b^4$.

19. $\dfrac{a^5+b^5}{a+b}=a^4-a^3b+a^2b^2-ab^3+b^4$.

20. $2(a-b)^2+5(a+b)+8\,ab=(2\,a+2\,b+1)(a+b+1)$.

Solve the following equations, and verify the results:

21. $(2\,a+3)(3\,a-2)=a^2+a(5\,a+3)$.

22. $6(b-4)^2=-5-(3-2\,b)^2-5(2+b)(7-2\,b)$.

23. $(y-3)^2+(y-4)^2-(y-2)^2-(y-3)^2=0$.

24. $(x-3)(3\,x+4)-(x-4)(x-2)=(2\,x+1)(x-6)$.

25. $2(3\,r-2)(4\,r+1)+(r-4)^3=(r+4)^3-2$.

26. $a^3-c+b^3c+abc=b$. (Solve for c.)

27. $(b-2)^2(b-y)-3\,by+(2\,b+1)(b-1)=3-2\,b$. (Find y.)

28. $2(12-x)+3(5\,x-4)+2(16-x)=12(3+x)$.

29. $(b-a)x-(a+b)x+4\,a^2=0$. (Find x.)

30. $(x-a)(b-c)+(b-a)(x-c)-(a-c)(x-b)=0$. (Find x.)

31. $r^3v+s^3v-3\,r-3\,s+3\,v(r^2s+rs^2)=0$. (Find v.)

32. $(x-3)(x-7)-(x-5)(x-2)+12=2(x-1)$.

33. $(a+b)^2+(x-b)(x-a)-(x+a)(x+b)=0$. (Find x.)

34. $ny(y+n)-(y+m)(y+n)(m+n)+my(y+m)=0$. (Find y.)

35. $(n+i)(j-i+k)-(n-i)(i-j+k)=0$. (Find n.)

36. $\frac{7}{12}(5x-1)+\frac{5}{16}(2-3x)+\frac{1}{3}(4+x)=\frac{3}{8}(1+2x)-\frac{9}{16}$.

37. $(l-m)(z-n)+2l(m+n)=(l+m)(z+n)$. (Find z.)

38. $a(x-b)-(a+b)(x+b-a)=b(x-a)+a^2-b^2$. (Find x.)

39. $(m+n)(a+b-y)+(n-m)(b-y)=n(m+b)$. (Find y.)

40. $\frac{3(2a-3b)}{8}-\frac{2(3a-5b)}{3}+\frac{5(a-b)}{6}=\frac{b}{8}$. (Find a.)

Solve each of the following equations for each letter in terms of the others.

41. $l(W+w')=l'W'$.

42. $(c-n)d=(c-n_1)d_1$.

43. $m_2s_2(t_2-t)=(m+m_1)(t-t_1)$.

44. $(m+m_1)(t_1-t)=lm_2+m_2t$.

PROBLEMS

1. What number must be added to each of the numbers 2, 26, 10 in order that the product of the first two sums may equal the square of the last sum?

2. What number must be subtracted from each of the numbers 9, 12, 18 in order that the product of the first two remainders may equal the square of the last remainder?

3. What number must be added to each of the numbers a, b, c in order that the product of the first two sums may equal the square of the last?

Note that problem 1 is a special case of 3. Explain how 2 may also be made a special case of 3.

4. What number must be added to each of the numbers a, b, c, d in order that the product of the first two sums may equal the product of the last two?

5. State and solve a problem which is a special case of problem 4.

6. What number must be added to each of the numbers a, b, c, d in order that the sum of the squares of the first two sums may equal the sum of the squares of the last two?

7. State and solve a problem which is a special case of problem 6.

8. What number must be added to each of the numbers a, b, c, d in order that the sum of the squares of the first two sums may be k more than twice the product of the last two?

9. State and solve a problem which is a special case of problem 8.

10. The radius of a circle is increased by 3 feet, thereby increasing the area of the circle by 50 square feet. Find the radius of the original circle.

The area of a circle is πr^2. Use $3\frac{1}{7}$ for π.

11. The radius of a circle is decreased by 2 feet, thereby decreasing the area by 36 square feet. Find the radius of the original circle.

12. State and solve a general problem of which 10 is a special case.

13. State and solve a general problem of which 11 is a special case.

How may the problem stated under 12 be interpreted so as to include the one given under 13?

14. Each side of a square is increased by a feet, thereby increasing its area by b square feet. Find the side of the original square.

Interpret this problem if a and b are both negative numbers.

15. State and solve a problem which is a special case of 14, (1) when a and b are both positive, (2) when a and b are both negative.

16. Two opposite sides of a square are each increased by a feet and the other two by b feet, thereby producing a rectangle whose area is c square feet greater than that of the square. Find the side of the square.

Interpret this problem when a, b, and c are all negative numbers.

17. State and solve a problem which is a special case of 16, (1) when a, b, and c are all positive, (2) when a, b, and c are all negative.

18. A messenger starts for a distant point at 4 A.M., going 5 miles per hour. Four hours later another starts from the same place, going in the same direction at the rate of 9 miles per hour. When will they be together? When will they be 8 miles apart? How far apart will they be at 2 P.M.?

For a general explanation of problems on motion, see p. 115, E. C.

19. One object moves with a velocity of v_1 feet per second and another along the same path in the same direction with a velocity of v_2 feet. If they start together, how long will it require the latter to gain n feet on the former?

From formula (2), p. 117, E. C., we have $t = \dfrac{n}{v_2 - v_1}$.

Discussion. If $v_2 > v_1$ and $n > 0$, the value of t is positive, *i.e.* the objects will be in the required position some time *after* the time of starting.

If $v_2 < v_1$ and $n > 0$, the value of t is negative, which may be taken to mean that if the objects had been moving before the instant taken in the problem as the time of starting, then they would have been in the required position some time *earlier*.

If $v_2 = v_1$ and $n \neq 0$, the solution is impossible. See § 25. This means that the objects will never be in the required position. If $v_1 = v_2$ and $n = 0$, the solution is indeterminate. See § 24. This may be interpreted to mean that the objects are always in the required position.

20. State and solve a problem which is a special case of 19 under each of the conditions mentioned in the discussion.

21. At what time after 5 o'clock are the hands of a clock first in a straight line?

22. Saturn completes its journey about the sun in 29 years and Uranus in 84 years. How many years elapse from conjunction to conjunction? See figure, p. 119, E. C.

23. An object moves in a fixed path at the rate of v_1 feet per second, and another which starts a seconds later moves in the same path at the rate of v_2 feet per second. In how many seconds will the latter overtake the former?

24. In problem 23 how long before they will be d feet apart?

If in problem 24 d is zero, this problem is the same as 23. If d is not zero and a is zero, it is the same as problem 19.

25. A beam carries 3 weights, one at each end weighing 100 and 120 pounds respectively, and the third weighing 150 pounds 2 feet from its center, where the fulcrum is. What is the length of the beam if this arrangement makes it balance?

For a general explanation of problems involving the lever, see pp. 120–122, E. C.

26. A beam whose fulcrum is at its center is made to balance when weights of 60 and 80 pounds are placed at one end and 2 feet from that end respectively, and weights of 50 and 100 pounds are placed at the other end and 3 feet from it respectively. Find the length of the beam.

27. How many cubic centimeters of matter, density 4.20, must be added to 150 ccm. of density 8.10 so that the density of the compound shall be 5.4? See § 99, E. C.

28. How many cubic centimeters of nitrogen, density 0.001255, must be mixed with 210 ccm. of oxygen, density 0.00143, to form air whose density is 0.001292?

29. A man can do a piece of work in 16 days, another in 18 days, and a third in 15 days. How many days will it require all to do it when working together?

30. A can do a piece of work in a days, B can do it in b days, C in c days, and D in d days. How long will it require all to do it when working together?

CHAPTER IV

INTEGRAL LINEAR EQUATIONS IN TWO OR MORE VARIABLES

INDETERMINATE EQUATIONS

65. If a single equation contains two unknowns, an **unlimited number** of pairs of numbers may be found which satisfy the equation.

E.g. In the equation, $y = 2x + 1$, by assigning any value to x, a corresponding value of y may be found such that the two together satisfy the equation.

Thus, $x = -3$, $y = -5$; $x = 0$, $y = 1$; $x = 2$, $y = 5$, are pairs of numbers which satisfy this equation.

For this reason a single equation in two unknowns is called an **indeterminate** equation, and the unknowns are called **variables.** A **solution** of such an equation is any pair of numbers which satisfy it.

A picture or map of the real (see §§ 135, 195) solutions of an indeterminate equation in two variables may be made by means of the **graph** as explained in §§ 107, 108, E. C.

66. The **degree** of an equation in two or more letters is the sum of the exponents of those letters in that one of its terms in which this sum is greatest. See § 110, E. C.

E.g. $y = 2x + 1$ is of the *first degree* in x and y. $y^2 = 2x + y$ and $y = 2xy + 3$ are each of the *second degree* in x and y.

An equation of the first degree in two variables is called a **linear** equation, since it can be shown that the graph of every such equation is a straight line.

67. It is often important to determine those solutions of an indeterminate equation which are **positive integers**, and for this purpose the graph is especially useful.

Ex. 1. Find the positive integral solutions of the equation

$$3x + 7y = 42.$$

Solution. Graph the equation carefully on cross-ruled paper, finding it to cut the x-axis at $x = 14$ and the y-axis at $y = 6$.

Look now for the *corner points* of the unit squares through which this straight line passes. The coördinates of these points, if there are such points, are the solutions required. In this case the line passes through only one such point, namely the point (7, 3). Hence the solution sought is $x = 7$, $y = 3$.

Ex. 2. Find the *least* positive integers which satisfy

$$7x - 3y = 17.$$

Solution. This line cuts the x-axis at $x = 2\frac{3}{7}$ and the y-axis at $y = -5\frac{2}{3}$. On locating these points as accurately as possible, the line through them *seems* to cut the corner points (5, 6) and (8, 13). The coördinates of both these points satisfy the equation. Hence the solution sought is $x = 5$, $y = 6$.

EXERCISES

Solve in positive integers by means of graphs, and check:

1. $x + y = 7$.
2. $x + y = 3$.
3. $x - 27 = -9y$.
4. $7y - 112 = -4x$.
5. $90 - 5x = 9y$.
6. $5x = 29 - 3y$.
7. $140 - 7x - 10y = 0$.
8. $8 - 2x - y = 0$.

Solve in least positive integers, and check:

9. $7x = 3y + 21$.
10. $5x - 4y = 20$.
11. $4x = 9y - 36$.
12. $5x - 2y + 10 = 0$.

68. In the case of two indeterminate equations, each of the first degree in two variables, the coördinates of the point of intersection of their graphs form a solution of *both equations*.

Since these graphs are straight lines, they have *only one point* in common, and hence there is only *one solution* of the given pair of equations.

E.g. On graphing $x + y = 4$ and $y - x = 2$, the lines are found to intersect in the point (1, 3). Hence the solution of this pair of equations is

$$x = 1,\ y = 3.$$

EXERCISES

Graph the following and thus find the solution of each pair of equations. Check by substituting in the equations.

1. $\begin{cases} 3x - 2y = -2, \\ x + 7y = 30. \end{cases}$

2. $\begin{cases} x + y = 2, \\ 3x + 2y = 3. \end{cases}$

3. $\begin{cases} x - 4y = 1, \\ 2x - y = -5. \end{cases}$

4. $\begin{cases} x = -1, \\ 2x - 3y = 1. \end{cases}$

5. $\begin{cases} 4x = 2y + 6, \\ x - 5 = y - 1. \end{cases}$

6. $\begin{cases} x = y - 5, \\ 5y = x + 9. \end{cases}$

7. $\begin{cases} 8x = 7y, \\ x + 3 = 5y + 3. \end{cases}$

8. $\begin{cases} y = 1, \\ 3y + 4x = y. \end{cases}$

9. $\begin{cases} 2x - 4y = 4, \\ x - y = 6y - 3. \end{cases}$

10. $\begin{cases} x = 4, \\ y + x = 8. \end{cases}$

11. $\begin{cases} y = -3, \\ 3x + 2y = 3. \end{cases}$

12. $\begin{cases} x = -2, \\ y = 5. \end{cases}$

SOLUTION BY ELIMINATION

69. The solution of a pair of equations such as the foregoing may be obtained without the use of the graph by the process called **elimination**. See pages 151–159, E. C.

70. Elimination by **substitution** consists in expressing one variable in terms of the other in one equation and substituting this result in the other equation, thus obtaining an equation in which only one variable appears. See § 116, E. C.

71. Elimination by **addition or subtraction** consists in making the coefficients of one variable the same in the two equations (§ 62), so that when the members are added or subtracted this variable will not appear in the resulting equation. See § 117, E. C.

72. Elimination by **comparison** is a third method, which consists in expressing the same variable in terms of the other in each equation and equating these two expressions to each other.

As an example of elimination by comparison, solve

$$\begin{cases} 3y + x = 14, & (1) \\ 2y - 5x = -19. & (2) \end{cases}$$

From (1), $x = 14 - 3y.$ (3)

From (2), $x = \dfrac{19 + 2y}{5}.$ (4)

From (3) and (4), $14 - 3y = \dfrac{19 + 2y}{5}.$ (5)

Solving (5), $y = 3.$

Substituting in (1), $x = 5.$

Check by substituting $x = 5$, $y = 3$ in both (1) and (2).

In applying any method of elimination it is desirable first to reduce each equation to the standard form: $ax + by = c$. See § 119, E. C.

EXERCISES

Solve the following pairs of equations by one of the processes of elimination.

1. $\begin{cases} 3x + 2y = 118, \\ x + 5y = 191. \end{cases}$

2. $\begin{cases} 5x - 8\frac{1}{2} = 7y - 44, \\ 2x = y + \frac{5}{7}. \end{cases}$

3. $\begin{cases} 6x - 3y = 7, \\ 2x - 2y = 3. \end{cases}$

4. $\begin{cases} 3x + 7y - 341 = 7\frac{1}{2}y + 43\frac{1}{2}x, \\ 2\frac{1}{2}x + \frac{1}{2}y = 1. \end{cases}$

5. $\begin{cases} 5x - 11y - 2 = 4x, \\ 5x - 2y = 63. \end{cases}$

6. $\begin{cases} 3y + 40 = 2x + 14, \\ 9y - 347 = 5x - 420. \end{cases}$

7. $\begin{cases} 5y - 3x + 8 = 4y + 2x + 7, \\ 4x - 2y = 3y + 2. \end{cases}$

8. $\begin{cases} 6y - 5x = 5x + 14, \\ 3y - 2x - 6 = 5 + x. \end{cases}$

9. $\begin{cases}(x+5)(y+7)=(x+1)(y-9)+112,\\ 2x+10=3y+1.\end{cases}$ 10. $\begin{cases}73-7y=5x,\\ 2y-3x=12.\end{cases}$

11. $\begin{cases}ax=by,\\ x+y=c.\end{cases}$ 13. $\begin{cases}x+y=a,\\ x-y=b.\end{cases}$ 15.* $\begin{cases}\frac{3}{x}-\frac{5}{y}=6,\\ \frac{2}{x}+\frac{3}{y}=2.\end{cases}$

12. $\begin{cases}\frac{x}{a}=\frac{y}{b},\\ x+y=s.\end{cases}$ 14. $\begin{cases}ax+by=c,\\ fx+gy=h.\end{cases}$ 16.* $\begin{cases}\frac{a}{x}+\frac{b}{y}=c,\\ \frac{f}{x}+\frac{g}{y}=h.\end{cases}$

SOLUTION BY FORMULA

73. We now proceed to a more general study of a pair of linear equations in two variables.

Ex. 1. Solve
$$\begin{cases}2x+3y=4, & (1)\\ 5x+6y=7. & (2)\end{cases}$$

Multiplying (1) by 5 and (2) by 2,
$$5\cdot 2x+5\cdot 3y=5\cdot 4, \quad (3)$$
$$2\cdot 5x+2\cdot 6y=2\cdot 7. \quad (4)$$

Subtracting (3) from (4), $(2\cdot 6-5\cdot 3)y=2\cdot 7-5\cdot 4.$ (5)

Solving for y, $$y=\frac{2\cdot 7-5\cdot 4}{2\cdot 6-5\cdot 3}=\frac{-6}{-3}=2. \quad (6)$$

In like manner, solving for x by eliminating y,

we have $$x=\frac{4\cdot 6-7\cdot 3}{2\cdot 6-5\cdot 3}=\frac{3}{-3}=-1. \quad (7)$$

Ex. 2. In this manner, solving,
$$\begin{cases}7x+9y=71,\\ 2x+3y=48,\end{cases}$$

we find $$x=\frac{71\cdot 3-48\cdot 9}{7\cdot 3-2\cdot 9} \text{ and } y=\frac{7\cdot 48-2\cdot 71}{7\cdot 3-2\cdot 9}.$$

* Let $\frac{1}{x}$ and $\frac{1}{y}$ be the unknowns.

In Ex. 2, the various coefficients are found to occupy the *same relative positions* in the expressions for x and y as the corresponding coefficients do in Ex. 1.

Show that this is also true in the following:

Ex. 3. $\begin{cases} 3x + 7y = 10, \\ 2x - 5y = 7. \end{cases}$ Ex. 4. $\begin{cases} 5x - 3y = 8, \\ 2x + 7y = 19. \end{cases}$

A convenient rule for reading directly the values of the unknowns in such a pair of equations may be made from the solution of the following:

Ex. 5. Solve $\begin{cases} a_1x + b_1y = c_1, \\ a_2x + b_2y = c_2. \end{cases}$

Eliminating first y and then x as in Ex. 1, we find:

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \text{ and } y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

To remember these results, notice that the coefficients of x and y in the given equations stand in the form of a square, thus $\begin{matrix} a_1 & b_1 \\ a_2 & b_2 \end{matrix}$, and that the denominator in the expressions for both x and y is the *cross product* a_1b_2 minus the *cross product* a_2b_1. The numerator in the expression for x is read by replacing the a's in this square by the c's, *i.e.*, $\begin{matrix} c_1 & b_1 \\ c_2 & b_2 \end{matrix}$, and then reading the cross products as before. The numerator for y is read by replacing the b's by the c's, *i.e.*, $\begin{matrix} a_1 & c_1 \\ a_2 & c_2 \end{matrix}$, and then reading the cross products.

74. To indicate that the coefficients in a pair of equations are to be treated as just described, we write $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \equiv a_1b_2 - a_2b_1$ and call $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ a **determinant**. These are much used in higher algebra.

Since any pair of linear equations in two unknowns may be reduced to the standard form as given in Ex. 5, it follows that the values of x and y there obtained constitute a *formula for the solution of any pair of such equations.*

EXERCISES

Reduce the following pairs of equations to the standard form and write out the solutions by the formula:

1. $\begin{cases} 3x + 4y = 10, \\ 4x + y = 9. \end{cases}$

2. $\begin{cases} 4x - 5y = -26, \\ 2x - 3y = -14. \end{cases}$

3. $\begin{cases} 6y - 17 = -5x, \\ 6x - 16 = -5y. \end{cases}$

4. $\begin{cases} \frac{1}{4}(x-3) = -\frac{1}{3}(y-2) + \frac{1}{2}x, \\ \frac{1}{2}(y-1) = x - \frac{1}{3}(x-2). \end{cases}$

5. $\begin{cases} 2x - y = 53, \\ 19x + 17y = 0. \end{cases}$

6. $\begin{cases} ax - by = 0, \\ x - y = c. \end{cases}$

7. $\begin{cases} mx + ny = p, \\ rx + sy = t. \end{cases}$

8. $\begin{cases} a(x+y) - b(x-y) = 2a, \\ a(x-y) - b(x+y) = 2b. \end{cases}$

9. $\begin{cases} (k+1)x + (k-2)y = 3a, \\ (k+3)x + (k-4)y = 7a. \end{cases}$

10. $\begin{cases} 2ax + 2by = 4a^2 + b^2, \\ x - 2y = 2a - b. \end{cases}$

11. $\begin{cases} (a+b)x - (a-b)y = 4ab, \\ (a-b)x + (a+b)y = 2a^2 - 2b^2. \end{cases}$

12. $\begin{cases} \frac{1}{2}(a-b) - \frac{1}{5}(a-3b) = b - 3, \\ \frac{3}{4}(a-b) + \frac{5}{6}(a+b) = 18. \end{cases}$

13. $\begin{cases} a(x+y) + b(x-y) = 2, \\ a^2(x+y) - b^2(x-y) = a - b. \end{cases}$

14. $\begin{cases} 7(x-5) = 3 - \dfrac{y}{2} - x, \\ \frac{1}{4}(x-y) + \frac{1}{2}y - \frac{5}{3}(x-1) = -1. \end{cases}$

15. $\begin{cases} mx + ny = m^3 + 2m^2n + n^3, \\ nx + my = m^3 + 2mn^2 + n^3. \end{cases}$

16. $\begin{cases} (m+n)x - (m-n)y = 2lm, \\ (m+l)x - (m-l)y = 2mn. \end{cases}$

17. $\begin{cases} \frac{1}{3}(5m - 7n + 2) - \frac{1}{4}(3m - 4n + 7) = n + 3\frac{5}{6}, \\ \frac{1}{4}(7m - 3n + 4) - \frac{1}{5}(6m - 5n + 7) = n - 2. \end{cases}$

18. $\begin{cases} (h+k)x + (h-k)y = 2(h^2 + k^2), \\ (h-k)x + (h+k)y = 2(h^2 - k^2). \end{cases}$

19. $\begin{cases} \frac{1}{2}(a+b-c)x + \frac{1}{2}(a-b+c)y = a^2 + (b-c)^2, \\ \frac{1}{2}(a-b+c)x + \frac{1}{2}(a+b-c)y = a^2 - (b-c)^2. \end{cases}$

INCONSISTENT AND DEPENDENT EQUATIONS

75. A pair of linear equations in two variables may be such that they either have no solution or have an unlimited number of solutions.

Ex. 1. Solve

$$\begin{cases} x - 2y = -2, & (1) \\ 3x - 6y = -12, & (2) \end{cases}$$

On graphing these equations they are found to represent two parallel lines. Since the lines have no point in common, it follows that the equations have no solution. See Fig. 1.

Attempting to solve them by means of the formula, § 73, we find:

$$x = \frac{(-2)(-6) - (-12)(-2)}{1(-6) - 3(-2)} = \frac{-12}{0},$$

and

$$y = \frac{1(-12) - 3(-2)}{1(-6) - 3(-2)} = \frac{-6}{0}.$$

But by § 25, $\frac{-12}{0}$ and $\frac{-6}{0}$ *are not numbers.* Hence, from this it follows that the given equations have no solution. In this case no solution is **possible**, and the equations are said to be **contradictory**.

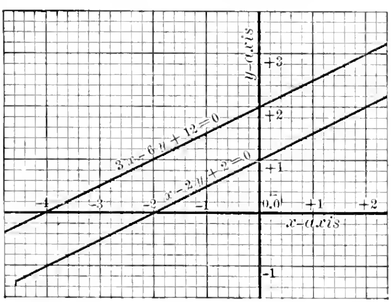

FIG. 1.

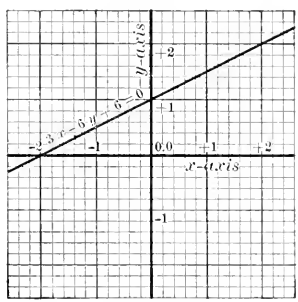

FIG. 2.

Ex. 2. Solve

$$\begin{cases} 3x - 6y = -6, & (1) \\ x - 2y = -2, & (2) \end{cases}$$

On graphing these equations, they are found to represent the *same line.* Hence every pair of numbers satisfying one equation must satisfy the other also. See Fig. 2.

Solving these equations by the formula, we find:

$$x = \frac{(-6)(-2)-(-2)(-6)}{3(-2)-1(-6)} = \frac{0}{0} \text{ and } y = \frac{3(-2)-1(-6)}{3(-2)-1(-6)} = \frac{0}{0}.$$

But by § 21, $\frac{0}{0}$ may represent *any number whatever*. Hence we may select for one of the unknowns any value we please and find from (1) or (2) a corresponding value for the other, but we may not select arbitrary values for *both* x and y.

In this case the solution is **indeterminate** and the equations are **dependent**; that is, one may be derived from the other.

Thus, (2) is derived from (1) by dividing both members by 3.

76. Two linear equations in two variables which have one and only one solution are called **independent and consistent**.

The cases in which such pairs of equations are *dependent* or *contradictory* are those in which the denominators of the expressions for x and y become *zero*. Hence, in order that such a pair of equations may have a *unique* solution, the denominator $a_1b_2 - a_2b_1$ of the formula, § 73, *must not reduce to zero*. This may be used as a test to determine whether a given pair of equations is independent and consistent.

EXERCISES

In the following, show both by the formula and by the graph which pairs of equations are independent and consistent, which dependent, and which contradictory.

1. $\begin{cases} 5x - 3y = 5, \\ 5x - 3y = 9. \end{cases}$

2. $\begin{cases} x - 7 + 5y = y - x - 2, \\ 5x + 3y - 4 = 3x - y + 3. \end{cases}$

3. $\begin{cases} 7x - 3y - 4 = 2x - 2, \\ x + y - 3 = 2x - 7. \end{cases}$

4. $\begin{cases} x - 3y = 6, \\ 5x - 15y = 18. \end{cases}$

5. $\begin{cases} 3y - 4x - 1 = 2x - 5y + 8, \\ 2y - 5x + 8 = 3x + y. \end{cases}$

6. $\begin{cases} 3x - 6y + 5 = 2x - 5y + 7, \\ 5x + 3y - 1 = 3x + 5y + 3. \end{cases}$

7. $\begin{cases} 2y + 7x = 2 + 6x, \\ 4x - 3y = 4 + 3x - 5y. \end{cases}$

8. $\begin{cases} 5x - 3 = 7y + 8, \\ 2x + 7 = 4y - 9. \end{cases}$

9. $\begin{cases} 5x + 2y = 6 + 3x + 5y, \\ 3x + y = 18 - 3x + 10y. \end{cases}$

10. $\begin{cases} 3x + 4y = 7 + 5y, \\ x - y = 6 - 2x. \end{cases}$

SYSTEMS OF EQUATIONS IN MORE THAN TWO VARIABLES

77. If a single linear equation in three or more variables is given, there is no limit to the number of sets of values which satisfy it.

E.g. $3x + 2y + 4z = 24$ is satisfied by $x = 1$, $y = 3$, $z = 3\frac{3}{4}$; $x = 2$, $y = 2$, $z = 3\frac{1}{2}$; $x = 0$, $y = 0$, $z = 6$; etc.

If two linear equations in three or more variables are given, they have in general an unlimited number of solutions.

E.g. $3x + 2y + 4z = 24$ and $x + y + z = 6$ are both satisfied by $x = 2$, $y = -1$, $z = 5$; $x = 3$, $y = -1\frac{1}{2}$, $z = 1\frac{1}{2}$; etc.

But if a system of linear equations contains *as many equations as variables*, it has in general one and only one set of values which satisfy all the equations.

E.g. The system $\begin{cases} x + y + z = 6, \\ 3x - y + 2z = 7, \\ 2x + 3y - z = 5, \end{cases}$

is satisfied by $x = 1$, $y = 2$, $z = 3$, and by no other set of values.

It may happen, however, as in the case of two variables, that such a system is not *independent* and *consistent.*

Such cases frequently occur in higher work, and a general rule is there found for determining the nature of a system of linear equations *without solving them:* namely, by means of *determinants* (§ 73). In this book the only test used is the result of the solution itself as explained in the next paragraph.

78. An independent and consistent system of linear equations in three variables may be solved as follows:

From two of the equations, say the 1st and 2d, eliminate one of the variables, obtaining *one* equation in the remaining two variables.

From the 1st and 3d equations eliminate the same variable, obtaining a *second* equation in the remaining two variables.

Solve as usual the two equations thus found. Substitute the values of these two variables in one of the given equations, and thus find the value of the third variable.

The process of elimination by addition or subtraction is usually most convenient. See § 120, E. C.

EXERCISES

Solve each of the following systems and check by substituting in each equation:

1. $\begin{cases} 2x + 5y - 7z = 9, \\ 5x - y + 3z = 16, \\ 7x + 6y + z = 34. \end{cases}$

2. $\begin{cases} a + b + c = 9, \\ 8a + 4b + 2c = 36, \\ 27a + 9b + 3c = 93. \end{cases}$

3. $\begin{cases} 18l - 7m - 5n = 161, \\ 4\frac{2}{5}m - \frac{2}{3}l + n = 18, \\ 3\frac{1}{2}n + 2m + \frac{3}{4}l = 33. \end{cases}$

4. $\begin{cases} \dfrac{a}{3} + \dfrac{b}{6} + \dfrac{c}{9} = -2, \\[2ex] \dfrac{a}{6} + \dfrac{b}{9} + \dfrac{c}{12} = -4, \\[2ex] \dfrac{a}{9} + \dfrac{b}{12} + \dfrac{c}{15} = -4. \end{cases}$

5. $\begin{cases} 8z - 3y + x = -2, \\ 3x - 5y - 6z = -46, \\ y + 5x - 4z = -18. \end{cases}$

6.* $\begin{cases} \dfrac{3}{a} = \dfrac{2}{b}, \\[2ex] \dfrac{2}{a} + \dfrac{5}{b} - \dfrac{4}{c} = 17, \\[2ex] \dfrac{7}{a} - \dfrac{3}{b} + \dfrac{6}{c} = 8. \end{cases}$

7. $\begin{cases} x + 2y - 3z = -3, \\ 2x - 3y + z = 8, \\ 5x - 4y - 7z = -5. \end{cases}$

8. $\begin{cases} 2x + 3y - 7z = 19, \\ 5x + 8y + 11z = 24, \\ 7x + 11y + 4z = 43. \end{cases}$

Show that this system is not independent.

9. $\begin{cases} x + y = 16, \\ z + x = 22, \\ y + z = 28. \end{cases}$

10. $\begin{cases} x + 2y = 26, \\ 3x + 4z = 56, \\ 5y + 6z = 65. \end{cases}$

11. $\begin{cases} a + b + c = 5, \\ 3a - 5b + 7c = 79, \\ 9a - 11b = 91. \end{cases}$

12. $\begin{cases} l + m + n = 29\frac{1}{4}, \\ l + m - n = 18\frac{1}{4}, \\ l - m + n = 13\frac{3}{4}. \end{cases}$

* Use $\frac{1}{a}$, $\frac{1}{b}$, and $\frac{1}{c}$ as the unknowns.

13. $\begin{cases} l+m+n=a, \\ l+m-n=b, \\ l-m+n=c. \end{cases}$

14. $\begin{cases} ax+by=p, \\ cy+dz=q, \\ ex+fz=r. \end{cases}$

15. $\begin{cases} \dfrac{1}{a}+\dfrac{1}{b}=4, \\ \dfrac{1}{a}+\dfrac{1}{c}=3, \\ \dfrac{1}{b}+\dfrac{1}{c}=2. \end{cases}$

16. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=a, \\ \dfrac{1}{x}+\dfrac{1}{z}=b, \\ \dfrac{1}{y}+\dfrac{1}{z}=c. \end{cases}$

17. $\begin{cases} u+2v+3x+4y=30, \\ 2u+3v+4x+5y=40, \\ 3u+4v+5x+6y=50, \\ 4u+5v+6x+7y=60. \end{cases}$

18. Make a rule for solving a system of four or more linear equations in as many variables as equations.

PROBLEMS INVOLVING TWO OR MORE UNKNOWNS

1. A man invests a certain amount of money at 4% interest and another amount at 5%, thereby obtaining an annual income of \$3100. If the first amount had been invested at 5% and the second at 4%, the income would have been \$3200. Find each investment.

2. The relation between the readings of the Centigrade and the Fahrenheit thermometers is given by the equation $F = 32 + \frac{9}{5}C$. The Fahrenheit reading at the melting temperature of osmium is 2432 degrees higher than the Centigrade. Find the melting temperature in each scale.

In the Réaumur thermometer the freezing and boiling points are marked 0° and 80° respectively. Hence if C is the Centigrade reading and R the Réaumur reading, then $R = \frac{4}{5}C$. See § 101, E. C.

3. What is the temperature Fahrenheit (*a*) if the Fahrenheit reading equals $\frac{1}{2}$ of the sum of the other two, (*b*) if the Centigrade reading equals $\frac{1}{2}$ of the Fahrenheit minus the Réaumur, (*c*) if the Réaumur is equal to the sum of the Fahrenheit and Centigrade?

4. Going with a current a steamer makes 19 miles per hour, while going against a current $\frac{3}{4}$ as strong the boat makes 5 miles per hour. Find the speed of each current and the boat.

5. There is a number consisting of 3 digits whose sum is 11. If the digits are written in reverse order, the resulting number is 594 less than the original number. Three times the tens' digit is one more than the sum of the hundreds' and the units' digit.

6. A certain kind of wine contains 20 % alcohol and another kind contains 28 %. How many gallons of each must be used to form 50 gallons of a mixture containing 21.6 % alcohol?

7. The area of a certain trapezoid of altitude 8 is 68. If 4 is added to the lower base and the upper base is doubled, the area is 108. Find both bases.

A trapezoid is a four-sided figure whose upper base, b_1, and lower base, b_2, are parallel, but the other two sides are not. If h is the perpendicular distance between the bases, then the area is $a = \frac{h}{2}(b_1 + b_2)$.

8. If on her second westward journey the *Lusitania* had made 1 knot more per hour, she would have crossed in 4 hours and 38 minutes less than she did. But if her speed had been 4 knots per hour less, she would have required 23 hours and 10 minutes longer. Find the time of her passage and her average speed if the length of her course was 2780 knots.

9. Aluminium bronze is an alloy of aluminium and copper. The densities of aluminium, copper, and aluminium bronze are 2.6, 8.9, and 7.69 respectively. How many ccm. of each metal are used in 100 ccm. of the alloy? See § 99, E. C.

10. Wood's metal, which is used in fire extinguishers on account of its low melting temperature, is an alloy of bismuth, lead, tin, and cadmium. In 120 pounds of Wood's metal, $\frac{1}{3}$ of the tin plus $\frac{1}{6}$ of the lead minus $\frac{1}{20}$ of the bismuth equals 7 pounds. If $\frac{1}{2}$ of the lead and $\frac{1}{5}$ of the tin be subtracted from the bismuth, the remainder is 42 pounds. Find the amount of each metal if 15 pounds of cadmium is used.

11. The upper base of a trapezoid is 6 and its area is 168. If $\frac{1}{3}$ the lower base is added to the upper, the area is 210. Find the altitude and the lower base.

12. A and B can do a piece of work in 18 days, B and C in 24 days, and C and A in 36 days. How long will it require each man, working alone, to do it, and how long will it require all working together?

13. A and B can do a piece of work in m days, B and C in n days, and C and A in p days. How long will it require each to do it working alone?

14. A beam resting on a fulcrum balances when it carries weights of 100 and 130 pounds at its respective ends. The beam will also balance if it carries weights of 80 and 110 pounds respectively 2 feet from the ends. Find the distance from the fulcrum to the ends of the beam.

15. A beam carries three weights, A, B, and C. A balance is obtained when A is 12 feet from the fulcrum, B 8 feet from the fulcrum (on the same side as A), and C 20 feet from the fulcrum (on the side opposite A). It also balances when the distance of A is 8 feet, B 10 feet, and C 18 feet. Find the weights B and C if A is 50 lbs.

16. At 0° Centigrade sound travels 1115 feet per second with the wind on a certain day, and 1065 feet per second against the wind. Find the velocity of sound in calm weather, and the velocity of the wind on this occasion.

17. If the velocities of sound in air, brass, and iron at 0° Centigrade are x, y, z meters per second respectively, then $3x + 2y - z = 2505$, $5x - 2y + z = 151$, and $x + y + z = 8777$. Find the velocity in each.

18. If x, y, z are the Centigrade readings at the temperatures which liquefy hydrogen, nitrogen, and oxygen respectively, then $3x - 8y + 2z = 410$, $-8x + 2y + 4z = 903$, and $x + 4y - 6z = 60$. Find each temperature in both Centigrade and Fahrenheit readings.

19. Two trapezoids have a common lower base. Their altitudes are 8 and 10 respectively, and the sum of their areas is 148. If the upper base of the first trapezoid is multiplied by 2 and that of the second divided by 2, their combined area is 152; while if the upper base of the first is divided by 2 and that of the second multiplied by 2, the combined area is 176. Find the bases of each trapezoid.

20. If x, y, z are the Centigrade readings at the freezing temperatures of hydrogen, nitrogen, and oxygen respectively, then we have $x+y-3z=199$, $2x-5y+z=328$, and $-4x+2y+2z=156$. Find each temperature.

21. If x, y, z are respectively the melting point of carbon, the temperature of the hydrogen flame in air, and the temperature of this flame in pure oxygen, then $10x+2y+z=41{,}892$, $15x+y+2z=60{,}212$, and $7x+y+z=29{,}368$. Find each.

22. If a, b, c are the values in millions of the mineral products of the United States in 1880, 1900, and 1906 respectively, find each from the following relations:

$$5a-\frac{b}{8}-\frac{c}{10}=1572,\quad \frac{a}{3}+\frac{b}{4}+\frac{c}{5}=669,\quad a-\frac{b}{2}+\frac{c}{7}=37.$$

23. If x, y, z represent in thousands of tons the steel products of the United States in 1880, 1890, and 1905, find each from the following relations:

$$x+y+z=25{,}547,\quad 3x+4y-z=826,\quad x-3y+z=8439.$$

24. If the number of millions of tons of coal mined in the United States in 1890, 1900, and 1906 be represented by x, y, z respectively, find each from the following relations:

$$\frac{x}{2}+\frac{y}{30}+\frac{z}{25}=105,\quad x-\frac{y}{9}+\frac{z}{17}=153,\quad 3x+2y-2z=164.$$

25. If the values in millions of the farm products of the United States in 1870, 1900, and 1906 are represented by l, m, and n respectively, find each from the following relations:

$$2l+m-n=1633,\quad 3l-2m+n=3440,\quad l+m+n=13{,}675.$$

26. The sum of the areas of two trapezoids whose altitudes are 10 and 12 respectively is 284. If the upper base of the first is multiplied by 3 while its lower is decreased by 2, and the upper base of the second is divided by 2 while its lower base is increased by 3, the sum of the areas is 382; if the upper bases of both are doubled and the lower bases of both divided by 2, the sum of the areas is 322; and if the upper bases are divided by 2 while the first lower is doubled and the second trebled, the sum of the areas is 388. Find the bases of each trapezoid.

27. Two boys carry a 120-pound weight by means of a pole, at a certain point of which the weight is hung. One boy holds the pole 5 ft. from the weight and the other 3 ft. from it. What proportion of the weight does each boy lift?

Solution. Let x and y be the required amounts, then $5x$ is the leverage of the first boy and $3y$ that of the second, and these must be equal as in the case of the teeter, p. 122, E. C. Hence we have

$$5x = 3y, \text{ and } x + y = 120.$$

Solving, we find $x = 45$, $y = 75$.

28. If, in problem 27, the boys lift P_1 and P_2 pounds respectively at distances d_1 and d_2, and w is the weight lifted, then

$$P_1 d_1 = P_2 d_2. \tag{1}$$

$$P_1 + P_2 = w. \tag{2}$$

Solve (1) and (2), (a) when P_1 and P_2 are unknown, (b) when P_1 and w are unknown, (c) when P_1 and d_2 are unknown.

29. A weight of 540 pounds is carried on a pole by two men at distances of 4 and 5 feet respectively. How much does each lift?

30. A weight of 470 pounds is carried by two men, one at a distance of 3 feet and the other lifting 200 pounds. At what distance is the latter?

31. Two men are carrying a weight on a pole at distances of 4 and 6 feet respectively. The former lifts 240 pounds. How many pounds are they carrying?

CHAPTER V

FACTORING

79. A rational integral expression is said to be **completely factored** when it cannot be further resolved into factors which are rational and integral. Such factors are called **prime factors.**

The simpler forms of factoring are given in the following outline.

A **monomial factor** of any expression is evident at sight, and its removal should be the first step in every case.

E.g. $$4\,ax^2 + 2\,a^2x = 2\,ax(2\,x + a).$$

FACTORS OF BINOMIALS

80. *The difference of two squares.*

E.g. $$4\,x^2 - 9\,z^4 = (2\,x + 3\,z^2)(2\,x - 3\,z^2).$$

81. *The difference of two cubes.*

E.g. $$\begin{aligned} 8\,x^3 - 27\,y^3 &= (2\,x - 3\,y)[(2\,x)^2 + (2\,x)(3\,y) + (3\,y)^2] \\ &= (2\,x - 3\,y)(4\,x^2 + 6\,xy + 9\,y^2). \end{aligned}$$

82. *The sum of two cubes.*

E.g. $$\begin{aligned} 27\,x^3 + 64\,y^3 &= (3\,x + 4\,y)[(3\,x)^2 - (3\,x)(4\,y) + (4\,y)^2] \\ &= (3\,x + 4\,y)(9\,x^2 - 12\,xy + 16\,y^2). \end{aligned}$$

FACTORS OF TRINOMIALS

83. *Trinomial squares.*

E.g. $$a^2 + 2\,ab + b^2 = (a + b)^2 = (a + b)(a + b),$$

and $$a^2 - 2\,ab + b^2 = (a - b)^2 = (a - b)(a - b).$$

84. *Trinomials of the form $x^2 + px + q$.*

E.g. $$x^2 + 3x - 10 = (x + 5)(x - 2).$$

A trinomial of this form has two binomial factors, $x + a$ and $x + b$, if two numbers a and b can be found whose product is q and whose algebraic sum is p.

85. *Trinomials of the form $mx^2 + nx + r$.*

E.g. $$6x^2 + 7x - 20 = (3x - 4)(2x + 5).$$

A trinomial of this form has two binomial factors of the type $ax + b$ and $cx + d$, if four numbers, a, b, c, d, can be found such that $ac = m$, $bd = r$, and $ad + bc = n$. See § 142, E. C.

86. *Trinomials which reduce to the difference of two squares.*

E.g. $$x^4 + x^2y^2 + y^4 = x^4 + 2x^2y^2 + y^4 - x^2y^2 = (x^2 + y^2)^2 - x^2y^2$$
$$= (x^2 + y^2 - xy)(x^2 + y^2 + xy).$$

In this case x^2y^2 is both added to and subtracted from the expression, whereby it becomes the difference of two squares. Evidently the term added and subtracted must itself be a *square*, and hence the degree of the trinomial must be 4 or a multiple of 4, since the degree of the middle term is *half* that of the trinomial.

Ex. $$4a^8 - 16a^4b^4 + 9b^8 = 4a^8 - 12a^4b^4 + 9b^8 - 4a^4b^4$$
$$= (2a^4 - 3b^4)^2 - 4a^4b^4$$
$$= (2a^4 - 3b^4 + 2a^2b^2)(2a^4 - 3b^4 - 2a^2b^2).$$

EXERCISES ON BINOMIALS AND TRINOMIALS

Factor the following:

1. $a^3 + b^3$.
2. $a^3 - b^3$.
3. $(a + b)^3 - c^3$.
4. $(a + b)^3 + c^3$.
5. $7ax^2 - 56a^4x^5$.
6. $a^5 - ab^4$.
7. $121x^7 - 4xy^4$.
8. $\frac{1}{8}a^3 + \frac{1}{125}b^3$.
9. $\frac{1}{8}r^3 - \frac{9}{32}rs^2$.
10. $8r^4 - 27r$.
11. $(a + b)^2 - c^2$.
12. $c^2 - (a - b)^2$.
13. $5c^2 + 7cd - 6d^2$.
14. $x^4 - 3x^2y^2 + y^4$.
15. $4x^2 - 12xy + 9y^2$.
16. $x^2 + 11xz + 30z^2$.

17. $6x^2 - 5xy - 6y^2$.

18. $3a^2x^2y^4 - 69a^2xy^2 + 336a^2$.

19. $20a^2b^2 + 23abx - 21x^2$.

20. $a^4 + 2a^2b^2 + 9b^4$.

21. $48a^3x^4y - 75ay^5$.

22. $16a^4x^3y + 54ay^4$.

23. $x^4y^2 + 2x^2yz + z^2$.

24. $a^2 + 10a - 39$.

25. $8a^2y^3 - 48a^2y^2z + 72a^2yz^2$.

26. $4m^8 - 60m^4n^4 + 81n^8$.

27. $35a^{2k} - 6a^kb^k - 9b^{2k}$.

28. $(a+b)^2 - (c-d)^2$.

29. $72a^2x^2 - 19axy^2 - 40y^4$.

30. $4(a-3)^6 - 37b^2(a-3)^3 + 9b^4$.

31. $6(x+y)^2 + 5(x^2 - y^2) - 6(x-y)^2$.

32. $9(x-a)^2 - 24(x-a)(x+a) + 16(x+a)^2$.

33. $12(c+d)^2 - 7(c+d)(c-d) - 12(c-d)^2$.

34. $(a^2 + 5a - 3)^2 - 25(a^2 + 5a - 3) + 150$.

FACTORS OF POLYNOMIALS OF FOUR TERMS

A polynomial of four terms may be readily factored in case it is in any one of the forms given in the next three paragraphs.

87. *It may be the cube of a binomial.*

Ex. 1. $a^3 - 3a^2b + 3ab^2 - b^3 = (a-b)^3$.

Ex. 2. $8x^3 + 36x^2y + 54xy^2 + 27y^3$

$= (2x)^3 + 3(2x)^2(3y) + 3(2x)(3y)^2 + (3y)^3$

$= (2x + 3y)^3$. See Ex. 31. (d). p. 23.

88. *It may be resolvable into the difference of two squares.*

In this case three of the terms must form a trinomial square.

Ex. 1. $a^2 - c^2 + 2ab + b^2 = (a^2 + 2ab + b^2) - c^2$

$= (a+b)^2 - c^2 = (a+b+c)(a+b-c)$.

Ex. 2. $4x^2 + z^6 - 4x^4 - 1 = z^6 - 4x^4 + 4x^2 - 1$

$= z^6 - (4x^4 - 4x^2 + 1) = z^6 - (2x^2 - 1)^2$

$= (z^3 + 2x^2 - 1)(z^3 - 2x^2 + 1)$.

89. *A binomial factor may be shown by grouping the terms.*

In this case the terms are grouped by twos as in the following examples.

Ex. 1. $ax + ay + bx^2 + bxy = (ax + ay) + (bx^2 + bxy)$
$= a(x + y) + bx(x + y) = (a + bx)(x + y).$

Ex. 2. $ax + bx + a^2 - b^2 = (ax + bx) + (a^2 - b^2)$
$= x(a + b) + (a - b)(a + b) = (x + a - b)(a + b).$

EXERCISES

Factor the following polynomials:

1. $x^3 + 3x^2y + 3xy^2 + y^3$.
2. $8a^3 - 36a^2b + 54ab^2 - 27b^3$.
3. $4a^4 - 4a^2b^2 + b^4 - 16x^2$.
4. $2ad + 3bc + 2ac + 3bd$.
5. $27x^3 - 54x^2y + 36xy^2 - 8y^3$.
6. $36a^4 - 24a^3 + 24a - 16$.
7. $mnx^2 - mrx - rn^2x + r^2n$.
8. $a^2b^2 - a^2bc^2n - abn + an^2c^2$.
9. $2y^2 + 4by + 3cy + 6bc$.
10. $bcyz + c^2z^2 + bdy + dcz$.
11. $5a^2c + 12cd - 6ad - 10ac^2$.
12. $a^2 - b^2x^2 + acx^2 - bcx^3$.
13. $b^3c^2 - c^2y^3 - b^3y^2 + y^5$.
14. $a^{3k} - 2a^{2k}b^k - 2a^kb^{2k} + b^{3k}$.
15. $m^{a+b} + m^an^a + m^bn^b + n^{a+b}$.
16. $b^2y^3 - b(c - d)y^2 + d(by - c) + d^2$.

FACTORS FOUND BY GROUPING

90. The discovery of factors by the proper grouping of terms as in § 89 is of wide application. Polynomials of five, six, or more terms may frequently be thus resolved into factors.

Ex. 1. $a^2 + 2ab + b^2 + 5a + 5b = (a + b)^2 + (5a + 5b)$
$= (a + b)(a + b) + 5(a + b) = (a + b + 5)(a + b).$

Ex. 2. $x^2 - 7x + 6 - ax + 6a = x^2 - 7x + 6 - (ax - 6a)$
$= (x - 1)(x - 6) - a(x - 6) = (x - 6)(x - 1 - a).$

Ex. 3. $a^2 - 2ab + b^2 - x^2 - 2xy - y^2$
$= (a^2 - 2ab + b^2) - (x^2 + 2xy + y^2)$
$= (a - b)^2 - (x - y)^2 = (a - b + x - y)(a - b - x + y).$

Ex. 4. $ax^2 + ax - 6a + x^2 + 7x + 12$
$= a(x^2 + x - 6) + (x^2 + 7x + 12)$
$= a(x + 3)(x - 2) + (x + 3)(x + 4)$
$= (x + 3)[a(x - 2) + x + 4] = (x + 3)(ax - 2a + x + 4).$

In some cases the grouping is effective only after a term has been separated into two parts.

Ex. 5. $2a^3 + 3a^2 + 3a + 1 = a^3 + (a^3 + 3a^2 + 3a + 1)$
$= a^3 + (a + 1)^3 = (a + a + 1)[a^2 - a(a + 1) + (a + 1)^2]$
$= (2a + 1)(a^2 + a + 1).$

As soon as the term $2a^3$ is separated into two terms the expression is shown to be the sum of two cubes.

Again, the grouping may be effective after a term has been both added and subtracted:

Ex. 6. $a^4 + b^4 = (a^4 + 2a^2b^2 + b^4) - 2a^2b^2$
$= (a^2 + b^2)^2 - (ab\sqrt{2})^2$
$= (a^2 + b^2 + ab\sqrt{2})(a^2 + b^2 - ab\sqrt{2}).$

In this case the factors are irrational as to one coefficient. Such factors are often useful in higher mathematical work.

EXERCISES

Factor the following:

1. $x^2 - 2xy + y^2 - ax + ay.$
2. $a^2 - ab + b^2 + a^3 + b^3.$
3. $a^3 - b^3 - a^2 - ab - b^2.$
4. $a^2 - 2ab + b^2 - x^2 + 2xy - y^2.$
5. $a^4 + 2a^3b - a^2c^2 + a^2b^2 - 2abc^2 - b^2c^2.$
6. $x^4 - y^4 + ax^2 + ay^2 - x^2 - y^2.$
7. $a^4 + a^2b^2 + b^4 + a^3 + b^3.$

In 7 group the first three and the last two terms.

8. $a^3 - 1 + 3x - 3x^2 + x^3.$ Group the last four terms.

9. $x^3 + x^2 + 3x + y^3 - y^2 + 3y.$

Group in pairs, the 1st and 4th, 2d and 5th, 3d and 6th terms.

10. $x^4 + x^3y - xy^3 - y^4 + x^3 - y^3.$

11. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 - x^4.$

12. $x^4 + 4x^2z - 4y^2 + 4yw + 4z^2 - w^2$.

13. $2a^2 - 12b^2 + 3bd - 5ab - 9bc - 6ac + 2ad$.

Group the terms: $2a^2 - 5ab - 12b^2$.

14. $a^2 + ab - 4ac - 2b^2 + 4bc + 3ad - 3bd$.

15. $a^3 + 2 - 3a$. Group thus: $(a^3 - 1) + (3 - 3a)$.

16. $4a^2 + a - 8ax - x + 4x^2$.

17. $3a^2 - 8ab + 4b^2 + 2ac - 4bc$.

18. $x^8 + y^8$, also $x^{16} + y^{16}$.

19. $a^6 + 2a^3b^3 + b^6 - 2a^4b - 2ab^4$.

20. $a^3 - 3a^2 + 4$. Group thus: $(a^3 - 2a^2) + (4 - a^2)$.

21. $a^2c - ac^2 - a^2b + ab^2 - b^2c + bc^2$.

22. $a^2b - a^2c + b^2c - ab^2 + ac^2 - bc^2$.

23. $3x^3 - x^2 - 4x + 2$. Add and subtract $-2x^2$.

24. $2x^3 - 11x^2 + 18x - 9$. Add and subtract $9x^2$.

FACTORS FOUND BY THE FACTOR THEOREM

91. It is possible to determine in advance whether a polynomial in x is divisible by a binomial of the form $x - a$.

E.g. In dividing $x^4 - 4x^3 + 7x^2 - 7x + 2$ by $x - 2$, the quotient is found to be $x^3 - 2x^2 + 3x - 1$.

Since *Quotient* × *Divisor* ≡ *Dividend*, we have

$$(x - 2)(x^3 - 2x^2 + 3x - 1) \equiv x^4 - 4x^3 + 7x^2 - 7x + 2.$$

As this is an *identity*, it holds for all values of x. For $x = 2$ the factor $(x - 2)$ is zero, and hence the left member is zero, § 22.

Hence for $x = 2$ the right member must also be zero. This is indeed the case, viz.:

$$2^4 - 4 \cdot 2^3 + 7 \cdot 2^2 - 7 \cdot 2 + 2 = 16 - 32 + 28 - 14 + 2 = 0.$$

Hence if $x - 2$ is a factor of $x^4 - 4x^3 + 7x^2 - 7x + 2$, the latter must reduce to zero for $x = 2$.

92. Theorem. *If a polynomial in* x *reduces to zero when a particular number* a *is substituted for* x, *then* $x - a$ *is a factor of the polynomial, and if the substitution of* a *for* x *does not reduce the polynomial to zero, then* $x - a$ *is not a factor.*

Proof. Let D represent any polynomial in x. Suppose D has been divided by $x - a$ until the remainder no longer contains x. Then, calling the quotient Q and the remainder R, we have the identity

$$D \equiv Q(x - a) + R, \tag{1}$$

which holds for *all* values of x.

The substitution of a for x in (1) does not affect R, reduces $Q(x - a)$ to zero, and may or may not reduce D to zero.

(1) If $x = a$ reduces D to zero, then $0 \equiv 0 + R$. Hence R is zero, and the division is exact. That is, $x - a$ is a factor of D.

(2) If $x = a$ does *not* reduce D to zero, then R is not zero, and the division is not exact. That is, $x - a$ is not a factor of D.

93. In applying the factor theorem the trial divisor must always be written in the form $x - a$.

Ex. 1. Factor $x^4 + 6x^3 + 3x^2 + x + 3$.

If there is a factor of the form $x - a$, then the only possible values of a are the various divisors of 3, namely $+1$, -1, $+3$, -3.

To test the factor $x + 1$, we write it in the form $x - (-1)$ where $a = -1$. Substituting -1 for x in the polynomial, we have

$$1 - 6 + 3 - 1 + 3 = 0.$$

Hence $x + 1$ is a factor.

On substituting $+1$, $+3$, -3 for x successively, no one reduces the polynomial to zero. Hence $x - 1$, $x - 3$, $x + 3$ are not factors.

Ex. 2. Factor $3x^3 - x^2 - 4x + 2$.

If $x - a$ is a factor, then a must be a factor of $+2$. We therefore substitute, $+2$, -2, $+1$, -1 and find the expression becomes zero when $+1$ is substituted for x. Hence $x - 1$ is a factor. The other factor is found by division to be $3x^2 + 2x - 2$, which is prime.

Hence $$3x^3 - x^2 - 4x + 2 = (x - 1)(3x^2 + 2x - 2).$$

EXERCISES

Factor by means of the factor theorem:

1. $3x^3 - 2x^2 + 5x - 6$.
2. $2x^3 + 3x^2 - 3x - 4$.
3. $2x^3 + x^2 - 12x + 9$.
4. $x^3 + 9x^2 + 10x + 2$.
5. $a^3 - 3a + 2$.
6. $m^3 + 5m^2 + 7m + 3$.
7. $x^4 + 3x^3 - 3x^2 - 7x + 6$.
8. $3r^3 + 5r^2 - 7r - 1$.
9. $2z^3 + 7z^2 + 4z + 3$.
10. $a^3 - 6a^2 + 11a - 6$.

11. Show by the factor theorem that $x^k - a^k$ contains the factor $x - a$ if k is *any* integer.

12. Show that $x^k - a^k$ contains the factor $x + a$ if k is any *even* integer.

13. Show that $x^k + a^k$ contains the factor $x + a$ if k is any *odd* integer.

14. Show that $x^k + a^k$ contains neither $x + a$ nor $x - a$ as a factor if k is an *even* integer.

MISCELLANEOUS EXERCISES ON FACTORING

1. $20a^3x^3y - 45a^3xy^3$.
2. $24am^5n^2 - 375am^2n^5$.
3. $432ar^4s + 54ars^4$.
4. $16x^2 - 72xy + 81y^2$.
5. $162a^3b + 252a^2b^2 + 98ab^3$.
6. $48x^5y - 12x^3y - 12x^2y + 3y$.
7. $12a^2bx^2 + 8ab^2x^2 + 18a^2bxy + 12ab^2xy$.
8. $18x^3y - 39x^2y^2 + 18xy^3$.
9. $4x^2 - 9xy + 6x - 9y + 4x + 6$.
10. $6x^2 - 13xy + 6y^2 - 3x + 2y$.
11. $6x^4 + 15x^2y^2 + 9y^4$.
12. $16x^4 + 24x^2y^2 + 8y^4$.
13. $15x^4 + 24x^2y^2 + 9y^4$.
14. $a^6 + y^6$.
15. $a^{12} + y^{12}$.
16. $a^8 - y^8$.
17. $a^{16} - y^{16}$.
18. $a^8 + a^4y^4 + y^8$.
19. $a^3 + a - 2$.
20. $a^8 - 18a^4y^4 + y^8$.
21. $a^{16} - 6a^8y^8 + y^{16}$.
22. $x^3 + 4x^2 + 2x - 1$.
23. $3x^3 + 2x^2 - 7x + 2$.

24. $a^8 - 3a^4y^4 + y^8$.

25. $a^3 + a^2 + a + 1$.

26. $a^3 + 9a^2 + 16a + 4$.

27. $2x^4 + x^3y + 2x^2y^2 + xy^3$.

28. $m^5 + m^4a + m^3a^2 + m^2a^3 + ma^4 + a^5$.

29. $(x-2)^3 - (y-z)^3$.

30. $a^6 + b^6 + 2ab(a^4 - a^2b^2 + b^4)$.

31. $x^5y^5 + x^4y^4z + x^3y^3z^2 + x^2y^2z^3 + xyz^4 + z^5$.

32. $8a^3 + 6ab(2a - 3b) - 27b^3$.

33. $a(x^3 + y^3) - ax(x^2 - y^2) - y^2(x + y)$.

34. $a^3 - b^3 + 3b^2c - 3bc^2 + c^3$.

35. $a^4 + 2a^3b - 2ab^2c - b^2c^2$.

36. $a^4 + 2a^3b + a^2b^2 - a^4b^2 - 2a^2b^2c - b^2c^2$.

SOLUTION OF EQUATIONS BY FACTORING

94. Many equations of higher degree than the first may be solved by factoring. (See §§ 144–146, E. C.)

Ex. 1. Solve $\quad 2x^3 - x^2 - 5x - 2 = 0. \quad (1)$

Factoring the left member of the equation, we have

$$(x-2)(x+1)(2x+1) = 0. \quad (2)$$

A value of x which makes one factor zero makes the whole left member zero and so satisfies the equation. Hence $x = 2$, $x = -1$, $x = -\frac{1}{2}$ are roots of the equation.

To solve an equation by this method first reduce it to the form $A = 0$, and then factor the left member. Put each factor equal to zero and solve for x. The results thus obtained are roots of the original equation.

Ex. 2. Solve $\quad x^3 - 12x^2 = 12 - 35x. \quad (1)$

Transposing and factoring, $(x-4)(x^2 - 8x + 3) = 0. \quad (2)$

Hence the roots of (1) are the roots of $x - 4 = 0$ and $x^2 - 8x + 3 = 0$. From $x - 4 = 0$, $x = 4$. The quadratic expression $x^2 - 8x + 3$ cannot be resolved into *rational* factors. See § 155.

EXERCISES

Solve each of the following equations by factoring, obtaining all roots which can be found by means of rational factors.

1. $x^3 + 3x^2 = 28x$.
2. $6x^3 + 8x + 5 = 19x^2$.
3. $x^4 + 12x^2 + 3 = 7x^3 + 9x$.
4. $12x^3 = 20x^2 + 5x + 6$.
5. $x^3 - 4x^2 = 4x + 5$.
6. $2x^3 + 3x = 9x^2 - 14$.
7. $5x^3 + x^2 - 14x + 8 = 0$.
8. $2x^3 + x^2 = 14x - 3$.
9. $12x^4 + 14x^3 + 1 = 3x^2 + 4x$.
10. $x^5 - 4x^4 - 40x^3 + 6 - x = 58x^2$.

COMMON FACTORS AND MULTIPLES

95. If each of two or more expressions is resolved into prime factors, then their **Highest Common Factor** (H. C. F.) is at once evident as in the following example. See § 182, E. C.

Given (1) $x^4 - y^4 = (x^2 + y^2)(x + y)(x - y)$,

(2) $x^6 - y^6 = (x^3 + y^3)(x^3 - y^3)$

$= (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2)$.

Then $(x + y)(x - y) = x^2 - y^2$ is the H. C. F. of (1) and (2).

In case only one of the given expressions can be factored by inspection, it is usually possible to select those of its factors, if any, which will divide the other expressions and so to determine the H. C. F.

Ex. Find the H. C. F. of $6x^3 + 4x^2 - 3x - 2$,
and $2x^4 + 2x^3 + x^2 - x - 1$.

By grouping we find:

$$6x^3 + 4x^2 - 3x - 2 = 2x^2(3x + 2) - (3x + 2)$$
$$= (2x^2 - 1)(3x + 2).$$

The other expression cannot readily be factored by any of the methods thus far studied. However, if there is a common factor, it must be either $2x^2 - 1$ or $3x + 2$. We see at once that it cannot be $3x + 2$. (Why?) By actual division $2x^2 - 1$ is found to be a factor of $2x^4 + 2x^3 + x^2 - x - 1$. Hence $2x^2 - 1$ is the H. C. F.

96. The **Lowest Common Multiple** (L. C. M.) of two or more expressions is readily found if these are resolved into prime factors. See § 185, E. C.

Ex. 1. Given $6\,abx - 6\,aby = 2 \cdot 3\,ab(x - y),$ (1)

$$8\,a^2x + 8\,a^2y = 2^3a^2(x + y), \tag{2}$$

$$36\,b^3(x^2 - y^2)(x + y) = 2^2 3^2 b^3(x - y)(x + y)^2. \tag{3}$$

The L. C. M. is $2^3 \cdot 3^2 a^2b^3 (x - y)(x + y)^2$, since this contains all the factors of (1), all the factors of (2) not found in (1), and all the factors of (3) not found in (1) and (2), with no factors to spare.

In case only one of the given expressions can be factored by inspection, it may be found by actual division whether or not any of these factors will divide the other expressions.

Ex. 2. Find the L. C. M. of $6\,x^3 - x^2 + 4\,x + 3,$ (1)

and $$6\,x^3 + 3\,x^2 - 10\,x - 5. \tag{2}$$

(1) is not readily factored. Grouping by twos, the factors of (2) are $3\,x^2 - 5$ and $2\,x + 1$. Now $3\,x^2 - 5$ is not a factor of (1). (Why?) Dividing (1) by $2\,x + 1$ the quotient is $3\,x^2 - 2\,x + 3$.

Hence $$6\,x^3 - x^2 + 4\,x + 3 = (2\,x + 1)(3\,x^2 - 2\,x + 3),$$

$$6\,x^3 + 3\,x^2 - 10\,x - 5 = (2\,x + 1)(3\,x^2 - 5).$$

Hence the L. C. M. is $(2\,x + 1)(3\,x^2 - 2\,x + 3)(3\,x^2 - 5)$.

Ex. 3. Find the L. C. M. of $a^3 + 2\,a^2 - a - 2,$ (1)

and $$10\,a^3 - 3\,a^2 + 4\,a + 1. \tag{2}$$

By means of the factor theorem, $a - 1$, $a + 1$, and $a + 2$ are found to be factors of (1), but none of the numbers, 1, -1, -2, when substituted for a in (2) will reduce it to zero. Hence (1) and (2) have no factors in common. The L. C. M. is therefore the product of the two expressions; viz. $(a + 1)(a - 1)(a + 2)(10\,a^3 - 3\,a^2 + 4\,a + 1)$.

97. The H. C. F. of three expressions may be obtained by finding the H. C. F. of two, and then the H. C. F. of this result and the third expression. Similarly the L. C. M. of three expressions may be obtained by finding the L. C. M. of two of them, and then the L. C. M. of this result and the third expression.

This may be extended to any number of expressions.

EXERCISES

Find the H. C. F. and also the L. C. M. in each of the following:

1. $x^2+y^2,\ x^6+y^6.$
2. $x^2+xy+y^2,\ x^3-y^3.$
3. $x^2-5x-6,\quad x^2-2x-3,\quad x^2+19x+18.$
4. $x^4-6x^2+1,\quad x^3+x^2-3x+1,\quad x^3+3x^2+x-1.$
5. $162a^3b+252a^2b^2+9ab^3,\quad 54a^3+42a^2b.$
6. $2x^3+x^2-8x+3,\quad x^2+2x-1.$
7. $3r^3+5r^2-7r-1,\quad 3r^2+8r+1.$
8. $a^3-3a^2+4,\quad ax-ab-2x+2b.$
9. $a^6+2a^3b^3+b^6-2a^4b-2ab^4,\quad a^3-2ab+b^3.$
10. $8a^3-36a^2b+54ab^2-27b^3,\quad 4a^2-9b^2.$
11. $36a^4-9a^2-24a-16,\quad 12a^3-6a^2-8a.$
12. $2y^2+4by+3cy+6bc,\quad y^2-3by-10b^2$
13. $x^{16}-y^{16},\quad x^8-y^8,\quad x^4-y^4.$
14. $m^3+8m^2+7m,\quad m^3+3m^2-m-3,\quad m^3-7m-6.$

98. An **important principle** relating to common factors is illustrated by the following example:

Given $\quad x^2+7x+10=(x+5)(x+2), \quad (1)$

and $\quad x^2-x-6=(x-3)(x+2). \quad (2)$

Add (1) and (2), $\quad 2x^2+6x+4=2(x+1)(x+2). \quad (3)$

Subtract (2) from (1), $\quad 8x+16=8(x+2). \quad (4)$

We observe that $x+2$, which is a common factor of (1) and (2), is also a factor of their sum (3), and of their difference (4). This example is a special case of the following:

99. Theorem 1. *A common factor of two expressions is also a factor of the sum or difference of any multiples of those expressions.*

Proof. Let A and B be any two expressions having the common factor f. Then if k and l are the remaining factors of A and B respectively,

$$A = fk \text{ and } B = fl.$$

Also let mA and nB be any multiples of A and B.

Then $mA = mfk$ and $nB = nfl$, from which we have:

$$mA \pm nB = mfk \pm nfl = f(mk \pm nl).$$

Hence f is a factor of $mA \pm nB$.

100. Theorem 2. *If* f *is a factor of* $mA \pm nB$ *and also of* A, *then* f *is a factor of* B, *provided* n *has no factor in common with* A.

Proof. Let f be a factor of $mA \pm nB$ and also of A, where mA and nB are integral multiples of the expressions A and B.

Then $\dfrac{mA \pm nB}{f}$, $\dfrac{A}{f}$, and $\dfrac{mA}{f}$ may each be reduced to an integral expression by cancellation.

Now $\dfrac{mA \pm nB}{f} = \dfrac{mA}{f} \pm \dfrac{nB}{f}$. Since $\dfrac{mA}{f}$ is integral, it follows that $\dfrac{nB}{f}$ is also integral. That is, f is a factor of nB. But f is not a factor of n since it is a factor of A, and by hypothesis n and A have no factor in common. Hence f is a factor of B.

101. By successive applications of the above theorems it is possible to find the H. C. F. of any two integral expressions.

Ex. 1. Find the H. C. F. of $9x^4 - x^2 + 2x - 1$, (1)

and $27x^5 + 8x^2 - 3x + 1$. (2)

Multiplying (1) by $3x$ and subtracting from (2) we have

$$\begin{array}{r} 27x^5 + 8x^2 - 3x + 1 \\ \underline{27x^5 - 3x^3 + 6x^2 - 3x} \\ 3x^3 + 2x^2 + 1 \end{array} \qquad (3)$$

By theorem **1**, any common factor of (1) and (2) is a factor of (3).

Calling expressions (1) and (2) B and A respectively of theorem **2**, then (3) is $A - 3x \cdot B$; and since the multiplier, $3x$, has no factor in common with (2), it follows from the theorem that any common factor of (3) and (2) is a factor of (1), and also that any common factor of (3) and (1) is a factor of (2). Hence (1) and (3) have the *same common factors*, that is, the same H. C. F. as (1) and (2). Therefore we proceed to obtain the H. C. F. of

$$9x^4 - x^2 + 2x - 1, \tag{1}$$

and

$$3x^3 + 2x^2 + 1. \tag{3}$$

Multiplying (3) by $3x$ and subtracting from (1) we have

$$-6x^3 - x^2 - x - 1. \tag{4}$$

By argument similar to that above, (3) and (4) have the same H. C. F. as (1) and (3) and hence the same as (1) and (2). Multiplying (3) by 2 and adding to (4) we have,

$$3x^2 - x + 1. \tag{5}$$

Then the H. C. F. of (5) and (3) is the same as that of (1) and (2). Multiplying (5) by x and subtracting from (3), we have

$$3x^2 - x + 1. \tag{6}$$

Then the H. C. F. of (5) and (6) is the same as that of (1) and (2). But (5) and (6) are identical, that is, their H. C. F. is $3x^2 - x + 1$. Hence this is the H. C. F. of (1) and (2).

The work may be conveniently arranged thus:

$$\begin{array}{llcrl}
(1) & 9x^4 \qquad - x^2 + 2x - 1 & & 27x^5 \qquad + 8x^2 - 3x + 1 & (2) \\
 & \underline{9x^4 + 6x^3 \qquad + 3x} & & \underline{27x^5 - 3x^3 + 6x^2 - 3x} & \\
(4) & -6x^3 - x^2 - x - 1 & & 3x^3 + 2x^2 \qquad + 1 & (3) \\
 & \underline{6x^3 + 4x^2 \qquad + 2} & & \underline{3x^3 - x^2 + x} & \\
(5) & 3x^2 - x + 1 & & 3x^2 - x + 1 & (6)
\end{array}$$

The object at each step is to obtain a new expression of as low a degree as possible. For this purpose the highest powers are eliminated step by step by the method of addition or subtraction.

E.g. In Ex. 1, x^5 was eliminated first, then x^4, and then x^3.

By theorems **1** and **2**, each new expression contains all the factors common to the given expressions. Hence, whenever an expression is reached which is *identical with the preceding one*, this is the H. C. F.

102. The process is further illustrated as follows:

Ex. 2. Find the H. C. F. of $2x^3 - 2x^2 - 3x - 2$,

and $3x^3 - x^2 - 2x - 16$.

Arranging the work as in Ex. 1, we have

$$
\begin{array}{lrrl}
(1) & 2x^3 - 2x^2 - 3x - 2 & 3x^3 - x^2 - 2x - 16 & (2) \\
 & 4x^3 - 4x^2 - 6x - 4 & 6x^3 - 2x^2 - 4x - 32 & \\
 & \underline{4x^3 + 5x^2 - 26x} & \underline{6x^3 - 6x^2 - 9x - 6} & \\
(4) & -9x^2 + 20x - 4 & 4x^2 + 5x - 26 & (3) \\
 & \underline{9x^2 - 18x} & 36x^2 + 45x - 234 & \\
(7) & 2x - 4 & \underline{-36x^2 + 80x - 16} & \\
(8) & x - 2 & 125x - 250 & (5) \\
 & & x - 2 & (6)
\end{array}
$$

Explanation. To eliminate x^3, we multiply (1) by 3 and (2) by 2 and subtract, obtaining (3).

To eliminate x^2 from (3), we need another expression of the second degree. To obtain this we multiply (1) by 2 and (3) by x and subtract, obtaining (4).

Using (4) and (3), we eliminate x^2, obtaining (5). Since (5) contains all factors common to (1) and (2), and since 125 is not such a factor, this is discarded without affecting the H. C. F., giving (6).

Multiplying (6) by 9 and adding to (4) we have (7). Discarding the factor 2 gives (8) which is identical with (6). Hence $x - 2$ is the H. C. F. sought.

103. Any *monomial* factors should be removed from each expression at the outset. If there are such factors *common* to the given expressions, these form a part of the H. C. F.

When this is done, then any monomial factor of any one of the derived expressions may be at once discarded without affecting the H. C. F., as in (5) of the preceding example.

In this way also the hypothesis of theorem **2** is always fulfilled; namely, that at every step the multiplier of one expression shall have no factor in common with the other expression.

Ex. 3. Find the H. C. F. of $3x^3 - 7x^2 + 3x - 2$,

and $x^4 - x^3 - x^2 - x - 2$.

$$
\begin{array}{lrl}
(1) & 3x^3 - 7x^2 + 3x - 2 & x^4 - x^3 - x^2 - x - 2 \quad (2) \\
 & 12x^3 - 28x^2 + 12x - 8 & 3x^4 - 3x^3 - 3x^2 - 3x - 6 \\
 & \underline{12x^3 - 18x^2 - 3x - 18} & \underline{3x^4 - 7x^3 + 3x^2 - 2x} \\
(4) & -10x^2 + 15x + 10 & 4x^3 - 6x^2 - x - 6 \quad (3) \\
(5) & -5(2x+1)(x-2). &
\end{array}
$$

Explanation. To eliminate x^4, we multiply (1) by x and (2) by 3 and subtract, obtaining (3).

To eliminate x^3, we multiply (1) by 4 and (3) by 3 and subtract, obtaining (4).

At this point the work may be shortened by factoring (4) as in (5). We may now reject, not only the factor -5, but also $2x + 1$, which is a factor of neither (1) nor (2), since $2x$ does not divide the highest power of either expression. But $x - 2$ is seen to be a factor of (2), by §§ 91, 92, and hence it is a common factor of (2) and (4) and therefore of (1) and (2). Hence $x - 2$ is the H. C. F. sought.

EXERCISES

Find the H.C.F. of the following pairs of expressions:

1. $a^3 + 6a^2 + 6a + 5,\ a^3 + 4a^2 - 4a + 5.$
2. $x^4 - 2x^3 - 2x^2 + 5x - 2,\ x^4 - 4x^3 + 6x^2 - 5x + 2.$
3. $2x^3 - 9x^2 - 13x - 4,\ x^3 - 12x^2 + 31x + 28.$
4. $x^4 - 5x^2 + 3x - 2,\ x^4 - 3x^3 + 3x^2 - 3x + 2.$
5. $2x^3 - 9x^2 + 8x - 2,\ 2x^3 + 5x^2 - 5x + 1.$
6. $3a^4 - 2a^3 + 10a^2 - 6a + 3,\ 2a^4 + 3a^3 + 5a^2 + 9a - 3.$
7. $15x^4 + 19x^3 - 44x^2 - 15x + 9,$

   $15x^4 - 6x^3 + 51x^2 + 11x - 15.$
8. $r^5 + 2r^4 - 2r^3 - 8r^2 - 7r - 2,\ r^5 - 2r^4 - 2r^3 + 4r^2 + r - 2.$

104. The following theorem enables us to find the L. C. M. of two expressions by means of the method which has just been used for finding the H. C. F.

Theorem 3. *The L.C.M. of two expressions is equal to the product of either expression and the quotient obtained by dividing the other by the H.C.F. of the two expressions.*

Proof. Let A and B be two expressions whose H.C.F. is F so that $A = mF$ and $B = nF$. Hence the L.C.M. of A and B is mnF. But $mnF = mnF = mB$. Also $mnF = nmF = nA$. Therefore the L.C.M. is either mB or nA, where $m = A \div F$ and $n = B \div F$.

Ex. Find the L.C.M. of $9x^4 - x^2 + 2x - 1$, (1)

and $27x^5 + 8x^2 - 3x + 1$. (2)

The H.C.F. was found in § 101 to be $3x^2 - x + 1$.
Dividing (1) by $3x^2 - x + 1$ we have $3x^2 + x - 1$.

Hence the L.C.M. of (1) and (2) is

$$(27x^5 + 8x^2 - 3x + 1)(3x^2 + x - 1).$$

EXERCISES

Find the L.C.M. of each of the following sets.

1. $a^4 + a^3 + 2a^2 - a + 3$, $a^4 + 2a^3 + 2a^2 - a + 4$.
2. $a^3 - 6a^2 + 11a - 6$, $a^3 - 9a^2 + 26a - 24$.
3. $2a^3 + 3a^2b - 2ab^2 - 3b^3$, $2a^4 - a^3b - 2a^2b^2 + 4ab^3 - 3b^4$.
4. $2a^3 - a^2b - 13ab^2 - 6b^3$,
$2a^4 - 5a^3b - 11a^2b^2 + 20ab^3 + 12b^4$.
5. $4a^3 - 15a^2 - 5a - 3$, $8a^4 - 34a^3 + 5a^2 - a + 3$,
$2a^3 - 7a^2 + 11a - 4$.
6. $a^4 + a^2 + 1$, $a^3 + 2a^2 - 2a + 3$.
7. $2k^3 - k^2l - 13kl^2 + 5l^3$, $3k^3 - 16k^2l + 24kl^2 - 7l^3$.
8. $12r^4 - 20r^3s - 15r^2s^2 + 35rs^3 - 12s^4$,
$6r^3 - 7r^2s - 11rs^2 + 12s^3$.
9. $2a^3 - 7a^2 + 6a - 2$, $a^3 + 2a^2 - 13a + 10$, $a^3 + 6a^2 + 6a + 5$.
10. $x^3 - xy^2 + yx^2 - y^3$, $2x^3 + x^2y + xy^2 + 2y^3$,
$2x^3 + 3x^2y + 3xy^2 + 2y^3$.

CHAPTER VI

POWERS AND ROOTS

105. Each of the operations thus far studied leads to a **single result.**

E.g. Two numbers have one and only one *sum*, § 2, and one and only one *product*, § 7.

When a number is subtracted from a given number, there is one and only one *remainder*, § 6.

When a number is divided by a given number, there is one and only one *quotient*, § 11.

We are now to study an operation which leads to **more than one result**; namely, the operation of finding roots.

Thus both 3 and -3 are square roots of 9, since $3 \cdot 3 = 9$, and also $(-3)(-3) = 9$; this is often indicated by $\sqrt{9} = \pm 3$. See § 114.

106. The operations of addition, subtraction, multiplication, and division are **possible** in all cases *except dividing by zero,* which is explicitly ruled out, §§ 24, 25.

Division is possible in general because *fractions* are admitted to the number system, and subtraction is possible in general because *negative numbers* are admitted. Thus $7 \div 3 = 2\frac{1}{3}$, $5 - 7 = -2$.

107. The operation of finding roots is not possible in all cases, unless other numbers besides positive and negative integers and fractions are admitted to the number system.

E.g. The number $\sqrt{2}$ is not an *integer* since $1^2 = 1$ and $2^2 = 4$.

Suppose $\sqrt{2} = \frac{a}{b}$ a fraction reduced to its lowest terms, so that a and b have no common factor. Then $\frac{a^2}{b^2} = 2$. But this is impossible, for if b^2 exactly divides a^2, then a and b must have factors in common. Hence $\sqrt{2}$ is not a *fraction*.

108. If a positive number is not the square of an integer or a fraction, a number may be found in terms of integers and fractions whose square differs from the given number by as little as we please. See p. 228, E. C.

E.g. 1.41, 1.414, 1.4141 are successive numbers whose squares differ by less and less from 2. In fact $(1.4141)^2$ differs from 2 by less than .0001, and by continuing the process by which these numbers are found, § 170, E. C., a number may be reached whose square differs from 2 by as little as we please.

1.41, 1.414, 1.4141, etc., are successive approximations to the number which we call *the square root of* 2, and which we represent by the symbol, $\sqrt{2}$.

109. Definition. If a number is not the kth power of an integer or a fraction, but if its kth root can be *approximated* by means of integers and fractions to any specified degree of accuracy, then such a kth root is called an **irrational number.** See § 36.

E.g. $\sqrt{2}$, $\sqrt[3]{2}$, $\sqrt[3]{5}$, etc., are irrational numbers, whereas $\sqrt{4}$, $\sqrt[3]{8}$, are rational numbers.

It is shown in higher algebra that irrational numbers correspond to definite points on the line of the number scale, § 49, E. C., just as integers and fractions do.

We, therefore, now enlarge the number system to include **irrational numbers** as well as integers and fractions.

It will be found also in higher work that there are other kinds of irrational numbers besides those here defined.

The set of numbers consisting of all rational and irrational numbers is called the **real number system.**

110. Even with the number system as thus enlarged, it is still not possible to find roots in all cases. The exception is the **even root of a negative number.**

E.g. $\sqrt{-4}$ is neither $+2$ nor -2, since $(+2)^2 = +4$ and $(-2)^2 = +4$, and no approximation to this root can be found as in the case of $\sqrt{2}$.

111. Definition. The indicated *even* root of a negative number, or any expression containing such a root, is called an **imaginary number**, or more properly, a **complex number.** All other numbers are called **real numbers.**

E.g. $\sqrt{-4}$, $\sqrt[4]{-2}$, $1+\sqrt{-2}$, are *complex numbers*, while 5, $\sqrt[3]{2}$, $1+\sqrt{2}$ are *real numbers*.

Complex numbers cannot be pictured on the line which represents real numbers, but another kind of graphic representation of complex numbers is made in higher algebraic work, and such numbers form the basis of some of the most important investigations in advanced mathematics.

112. With the number system thus enlarged, by the admission of irrational and complex numbers, we have the following **fundamental definition.**

$$(\sqrt[k]{n})^k = n.$$

That is, a kth root of any number n is such a number that, if it be raised to the kth power, the result is n itself.

E.g. $(\sqrt[3]{2})^3 = 2$, $(\sqrt{1})^2 = 1$, $(\sqrt{-2})^2 = -2$.

The imaginary or complex **unit** is $\sqrt{-1}$. By the above definition we have

$$(\sqrt{-1})^2 = -1.$$

In operating upon complex numbers, they should first be expressed in terms of the **imaginary unit.**

E.g. $\sqrt{-2} = \sqrt{2}\cdot\sqrt{-1}$, $\sqrt{-16} = \sqrt{16}\cdot\sqrt{-1} = 4\sqrt{-1}$.

$$\sqrt{-4}\cdot\sqrt{-9} = (\sqrt{4}\cdot\sqrt{-1})(\sqrt{9}\cdot\sqrt{-1}) = 2\cdot 3(\sqrt{-1})^2 = -6.$$

$$\sqrt{-4}+\sqrt{-9} = \sqrt{4}\cdot\sqrt{-1}+\sqrt{9}\cdot\sqrt{-1} = (2+3)\sqrt{-1} = 5\sqrt{-1}.$$

$$\frac{\sqrt{-16}}{\sqrt{-9}} = \frac{\sqrt{16}\cdot\sqrt{-1}}{\sqrt{9}\cdot\sqrt{-1}} = \frac{\sqrt{16}}{\sqrt{9}} = \frac{4}{3}.$$

113. By means of irrational and complex numbers it can be shown that every number has *two* square roots, *three* cube roots, *four* fourth roots, etc. See § 195, Ex. 17–20.

E.g. The square roots of 9 are $+3$ and -3. The square roots of -9 are $\pm\sqrt{-9} = \pm 3\sqrt{-1}$. The cube roots of 8 are 2, $-1+\sqrt{-3}$ and $-1-\sqrt{-3}$. The fourth roots of 16 are $+2$, -2, $+2\sqrt{-1}$ and $-2\sqrt{-1}$.

Any positive real number has two real roots of *even* degree, one positive and one negative.

E.g. $\sqrt[4]{16} = \pm 2$. The square roots of 3 are $\pm\sqrt{3}$.

Any real number, positive or negative, has one real root of *odd* degree, whose sign is the same as that of the number itself.

E.g. $\sqrt[3]{27} = 3$ and $\sqrt[5]{-32} = -2$.

114. The positive *even* root of a positive real number, or the real *odd* root of any real number, is called the **principal root**.

The positive square root of a negative real number is also sometimes called the *principal* imaginary root.

E.g. 2 is the principal square root of 4, 3 is the principal 4th root of 81; -4 is the principal cube root of -64; and $+\sqrt{-3}$ is the principal square root of -3.

Unless otherwise stated the radical sign is understood to indicate the *principal root.*

The only exception in this book is in such cases as, $\sqrt{4} = \pm 2$, where it represents *either* square root. But in such expressions as $1+\sqrt{2}$, $3\pm\sqrt{6}$, etc., the *principal root* only is understood.

In all cases it is easily seen from the context in what sense the sign is used.

When it is desired to designate in *particular* the principal root, the symbol $\sqrt{\quad}$ is used.

E.g. $\sqrt[4]{16} = 2$, while $\sqrt[4]{16}$ might stand indifferently for 2, -2, $2\sqrt{-1}$, or $-2\sqrt{-1}$.

$\sqrt[3]{8} = 2$, while $\sqrt[3]{8}$ might represent 2, $-1+\sqrt{-3}$, or $-1-\sqrt{-3}$.

THEOREMS ON POWERS AND ROOTS

115. Theorem 1. *The **nth** power of the **kth** power of any base is the **nkth** power of that base.*

Proof. Let n and k be any positive integers and let b be any base.

Then $(b^k)^n = b^k \cdot b^k \cdot b^k \cdots$ to n factors. § 124, E. C.

$= b^{k+k+k \cdots \text{ to } n \text{ terms}} = b^{nk}$. § 43

Hence $(b^k)^n = b^{nk}$.

Corollary $(b^k)^n = (b^n)^k = b^{nk}$.

E.g. $(2^3)^2 = (2^2)^3 = 2^6 = 64$.

116. Theorem 2. *The **nth** power of the product of several factors is the product of the **nth** powers of those factors.*

Proof. Let k, r, and n be any positive integers. Then

$(a^k b^r)^n = (a^k b^r) \cdot (a^k b^r) \cdots$ to n factors. § 124, E.C.

$= (a^k \cdot a^k \cdots$ to n factors$)(b^k \cdot b^k \cdots$ to n factors$)$ §§ 8, 9

$= (a^k)^n \cdot (b^r)^n$, § 121, E.C.

Hence, $(a^k b^r)^n = a^{nk} b^{nr}$. § 115

E.g. $(2^3 \cdot 3^2)^2 = 2^6 \cdot 3^4$.

117. Theorem 3. *The **nth** power of the quotient of two numbers equals the quotient of the **nth** powers of those numbers.*

Proof. We have $\left(\frac{a^k}{b^r}\right)^n = \frac{a^k}{b^r} \cdot \frac{a^k}{b^r} \cdot \frac{a^k}{b^r} \cdots$ to n factors § 124, E.C.

$= \frac{a^k \cdot a^k \cdot a^k \cdots \text{ to } n \text{ factors}}{b^r \cdot b^r \cdot b^r \cdots \text{ to } n \text{ factors}}$. § 193, E.C.

Hence, $\left(\frac{a^k}{b^r}\right)^n = \frac{a^{nk}}{b^{nr}}$. § 115

E.g. $\left(\frac{2^3}{3^2}\right)^2 = \frac{2^6}{3^4} = \frac{64}{81}$.

118. It follows from theorems **1**, **2**, and **3** that:

Any positive integral power of a monomial is found by multiplying the exponents of the factors by the exponent of the power.

119. Theorem 4. *The principal* r*th root of the* kr*th power of any positive real number is a power of that number whose exponent is* $kr \div r = k$.

Proof. Let k and r be positive integers and let b be any positive real number.

We are to prove that $\sqrt[r]{b^{kr}} = b^k$.

From theorem **1**, $(b^k)^r = b^{kr}$.

Hence by definition b^k is an rth root of b^{kr}, and since b^k is *real* and *positive*, it is the *principal* rth root of b^{kr} (§ 114).

That is, $\sqrt[r]{b^{kr}} = b^{kr \div r} = b^k$.

E.g. $\sqrt{3^4} = 3^{4 \div 2} = 3^2 = 9$; $\sqrt[4]{2^{12}} = 2^{12 \div 4} = 2^3 = 8$.

But it does *not* follow that

$$\sqrt[4]{(-2)^{12}} = (-2)^{12 \div 4} = (-2)^3 = -8,$$

since $(-2)^{12} = (2)^{12}$ and hence $\sqrt[4]{(-2)^{12}} = \sqrt[4]{2^{12}} = +8$.

The corresponding theorem holds when b is *negative* if r is *odd* and also when b is *negative* if k is *even*.

E.g. $\sqrt[3]{(-2)^6} = (-2)^{6 \div 3} = (-2)^2 = 4$; $\sqrt[3]{(-2)^{15}} = (-2)^5 = -32$.

120. Theorem 5. *The principal* r*th root of the product of two positive real numbers equals the product of the principal* r*th roots of the number.*

Proof. Let a and b be any positive real numbers and let r be any positive integer.

We are to prove $\sqrt[r]{ab} = \sqrt[r]{a} \cdot \sqrt[r]{b}$.

We have $(\sqrt[r]{a} \cdot \sqrt[r]{b})^r = (\sqrt[r]{a})^r \cdot (\sqrt[r]{b})^r$ § 116

$= a \cdot b.$ § 112

Hence, $ab = (\sqrt[r]{a} \cdot \sqrt[r]{b})^r$. § 3

Taking the principal rth root of both members,

we have $$\sqrt[r]{ab} = \sqrt[r]{a} \cdot \sqrt[r]{b}.$$

When r is *even* the corresponding theorem does not hold if a and b are both *negative*.

For example, it is *not true* that $\sqrt{(-4)(-9)} = \sqrt{-4} \cdot \sqrt{-9}$. For $\sqrt{(-4)(-9)} = \sqrt{36} = 6$; while $\sqrt{-4} \cdot \sqrt{-9}$
$$= 2\sqrt{-1} \cdot 3\sqrt{-1} = 6(\sqrt{-1})^2 = -6. \qquad \text{See § 112}$$

121. Theorem 6. *The principal* **rth** *root of the quotient of two positive real numbers equals the quotient of the principal* **rth** *roots of the numbers.*

Proof. Let a and b be any positive real numbers and let r be any positive integer.

We are to prove $$\sqrt[r]{\frac{a}{b}} = \frac{\sqrt[r]{a}}{\sqrt[r]{b}}.$$

We have $$\left(\frac{\sqrt[r]{a}}{\sqrt[r]{b}}\right)^r = \frac{(\sqrt[r]{a})^r}{(\sqrt[r]{b})^r} = \frac{a}{b}. \qquad \text{§§ 117, 112}$$

Hence, taking the principal rth root of both members,

we have $$\sqrt[r]{\frac{a}{b}} = \frac{\sqrt[r]{a}}{\sqrt[r]{b}}.$$

E.g. $$\sqrt{\frac{16}{25}} = \frac{\sqrt{16}}{\sqrt{25}} = \frac{4}{5}; \quad \sqrt[3]{\frac{-8}{27}} = \frac{\sqrt[3]{-8}}{\sqrt[3]{27}} = \frac{-2}{3} = -\frac{2}{3}.$$

The corresponding theorem does *not* hold when r is *even* if a is positive and b is negative. Thus it is not true that

$$\sqrt{\frac{4}{-9}} = \frac{\sqrt{4}}{\sqrt{-9}} = \frac{2}{3\sqrt{-1}} = \frac{2\sqrt{-1}}{3(\sqrt{-1})^2} = \frac{2\sqrt{-1}}{-3} = -\frac{2}{3}\sqrt{-1}.$$

But we have $$\sqrt{\frac{4}{-9}} = \sqrt{\frac{4}{9}(-1)} = \frac{2}{3}\sqrt{-1}.$$

If r is odd, the theorem holds for *all* real values of a and b.

122. From theorems **4**, **5**, **6**, it follows that:

If a monomial is a perfect power of the kth degree, its kth root may be found by dividing the exponent of each factor by the index of the root.

In applying the above theorems to the reduction of algebraic expressions containing letters, it is assumed that the values of the letters are such that the theorems apply.

EXERCISES

Find the following indicated powers and roots, and reduce each expression to its simplest form:

1. $(a^3b^4c^5)^7$.

2. $(2^{a+b}\cdot 3^c\cdot 5^b)^{a-b}$.

3. $\left(\dfrac{a^3b^4c^5}{2^3\cdot 3^2\cdot 4^3}\right)^2$.

4. $(a^{x-y})^{x^2+xy+y^2}$.

5. $(x^4y^5z^{x+y})^{x-y}$.

6. $\left(\dfrac{5^2b^2mn}{3^7bc^4}\right)^3$.

7. $(3^5\cdot 4^5\cdot 2^5)^{a-b}$.

8. $\sqrt[a]{3^{2a}\cdot 2^a\cdot 5^{3a}}$.

9. $\sqrt[3]{\dfrac{-27\cdot 8a^6}{64c^9d^{6a}}}$.

10. $(a^{m+n-1}b^{m-n}c^{mn})^{m+n}$.

11. $(3^{a+4}\cdot 4^{b-7}\cdot 5^{c-1})^{abc}$.

12. $\sqrt[2a]{3^{6a}\cdot 4^{2a}\cdot 5^{8a}\cdot 7^{4a}}$.

13. $\sqrt[a-b]{3^{a^2-b^2}\cdot 4^{a-b}\cdot 5^{a^2-b^2}}$.

14. $\sqrt{64\cdot 25\cdot 256\cdot 625}$.

15. $\sqrt[3]{27\cdot 125\cdot 64\cdot 3^{6a}}$.

16. $(a-b)^{m-n}(b-c)^{m-n}(a+b)^{m-n}$.

17. $\sqrt{\dfrac{(a-b)^2(a^2+2ab+b^2)}{(a-b)^4(a+b)^2}}$.

18. $\sqrt{\dfrac{(4x^2+4x+1)(4x^2-4x+1)}{36x^4-12x^2+1}}$.

19. $\sqrt[3]{(-343)(-27)x^6(a+b)^{3a}}$.

20. $\sqrt[3]{\dfrac{(-8)(-27)(-125)a^{3m}b^{9n}}{(-1)(-512)(1000)x^{15a}y^{21}}}$.

ROOTS OF POLYNOMIALS

123. In the Elementary Course, pp. 221–224, it was shown that the process for finding the **square root** of a polynomial is obtained by studying the relation of the square, $a^2 + 2\,ab + b^2$, to its square root, $a+b$.

In like manner the process for finding the **cube root** of a polynomial is obtained by studying the relation of the cube, $a^3 + 3\,a^2b + 3\,ab^2 + b^3$ or $a^3 + b(3\,a^2 + 3\,ab + b^2)$, to its cube root, $a + b$.

An example will illustrate the process.

Ex. 1. Find the cube root of

$$27\,m^3 + 108\,m^2n + 144\,mn^2 + 64\,n^3.$$

Given cube,	$27\,m^3+108\,m^2n+144\,mn^2+64\,n^3$	$3\,m+4\,n$, cube root
$a^3 = 27\,m^3$		1st partial product
$3\,a^2 = 27\,m^2$	$108\,m^2n+144\,mn^2+64\,n^3$,	1st remainder
$3\,ab = 36\,mn$		
$b^2 = 16\,n^2$		
$3\,a^2+3\,ab+b^2=27\,m^2+36\,mn+16\,n^2$	$108\,m^2n+144\,mn^2+64\,n^3$	$= b(3\,a^2+3\,ab+b^2)$
	0	

Explanation. The cube root of the first term, namely $3\,m$, is the first term of the root and corresponds to a of the formula. Cubing $3\,m$ gives $27\,m^3$ which is the a^3 of the formula.

Subtracting $27\,m^3$ leaves $108\,m^2n + 144\,mn^2 + 64\,n^3$, which is the $b(3\,a^2 + 3\,ab + b^2)$ of the formula.

Since b is not yet known, we cannot find completely either factor of $b(3\,a^2 + 3\,ab + b^2)$, but since a has been found, we can get the first term of the factor $3\,a^2+3\,ab+b^2$; viz. $3\,a^2$ or $3(3\,m)^2 = 27\,m^2$, which is the partial divisor. Dividing $108\,m^2n$ by $27\,m^2$ we have $4\,n$, which is the b of the formula.

Then $3\,a^2 + 3\,ab + b^2 = 3(3\,m)^2 + 3(3\,m)(4\,n) + (4\,n)^2 = 27\,m^2 + 36\,mn + 16\,n^2$ is the complete divisor. This expression is then multiplied by $b = 4\,n$, giving $108\,m^2n + 144\,mn^2 + 64\,n^3$, which corresponds to $b(3\,a^2 + 3\,ab + b^2)$ of the formula. On subtracting, the remainder is zero and the process ends. Hence, $3\,m + 4\,n$ is the required root.

Ex. 2. Find the cube root of

$$33x^4 - 9x^5 + x^6 - 63x^3 + 66x^2 - 36x + 8.$$

We first arrange the terms with respect to the exponents of x.

	$x^2 - 3x + 2$, cube root
Given cube,	$x^6 - 9x^5 + 33x^4 - 63x^3 + 66x^2 - 36x + 8$
	$a^3 = x^6$
$3a^2 = 3x^4$	$-9x^5 + 33x^4 - 63x^3 + 66x^2 - 36x + 8$
$3a^2 + 3ab + b^2 = 3x^4 - 9x^3 + 9x^2$	$-9x^5 + 27x^4 - 27x^3$
$3a'^2 = 3(x^2 - 3x)^2 = 3x^4 - 18x^3 + 27x^2$	$6x^4 - 36x^3 + 66x^2 - 36x + 8$
$3a'^2 + 3a'b' + b'^2 = 3x^4 - 18x^3 + 33x^2 - 18x + 4$	$6x^4 - 36x^3 + 66x^2 - 36x + 8$
	0

The cube root of x^6, or x^2, is the first term of the root. The first partial divisor, which corresponds to $3a^2$ of the formula, is $3(x^2)^2 = 3x^4$. Dividing $-9x^5$ by $3x^4$ we have $-3x$, which is the second term of the quotient, corresponding to b of the formula.

After these two terms of the root have been found, we consider $x^2 - 3x$ as the a of the formula and call it a'. The new partial divisor is $3a'^2 = 3(x^2 - 3x)^2 = 3x^4 - 18x^3 + 27x^2$, and the new b, which we call b', is then found to be 2.

Substituting $x^2 - 3x$ for a' and 2 for b' in $3a'^2 + 3a'b' + b'^2$, we have $3x^4 - 18x^3 + 33x^2 - 18x + 4$, which is the complete divisor. On multiplying this expression by 2 and subtracting, the remainder is zero. Hence the root is $x^2 - 3x + 2$.

In case there are four terms in the root, the sum of the first three, when found as above, is regarded as the new a, called a''. The remaining term is the new b and is called b''. The process is then precisely the same as in the preceding step.

EXERCISES

Find the square roots of the following:

1. $m^2 + 4mn + 6ml + 4n^2 + 12ln + 9l^2$.

2. $4x^4 + 8ax^3 + 4a^2x^2 + 16b^2x^2 + 16ab^2x + 16b^4$.

3. $9a^2 - 6ab + 30ac + 6ad + b^2 - 10bc - 2bd + 25c^2 + 10cd + d^2$.

4. $9a^2 - 30ab - 3ab^2 + 25b^2 + 5b^3 + \frac{b^4}{4}$.

5. $\frac{4}{9}a^2x^4 - \frac{4}{3}abx^3z + \frac{8}{3}a^2bx^2z^2 + b^2x^2z^2 - 4ab^2xz^3 + 4a^2b^2z^4$.

6. $a^2 - 6ab + 10ac - 14ad + 9b^2 - 30bc + 42bd + 25c^2$ $[- 70cd + 49d^2$.

7. $\frac{9}{4} + 6x - 17x^2 - 28x^3 + 49x^4$.

8. $9a^6 - 24a^3b^4 - 18a^3c^5 + 6a^3d^2 + 16b^8 + 24b^4c^5 - 8b^4d^2$ $[+ 9c^{10} - 6c^5d^2 + d^4$.

9. $25a^{4m}b^{6k} - 70a^{5m}b^{5k} + 49a^{6m}b^{4k}$.

10. $x^{10} - 8x^8w^5 + 16w^{10} - 4x^5y^3 + 16y^3w^5 + 4y^6 + 6x^5z^4$ $- 24z^4w^5 - 12y^3z^4 + 9z^8$.

Find the cube root of each of the following:

11. $x^3 - 3x^2y + 3xy^2 - y^3 + 3x^2z - 6xyz + 3y^2z + 3xz^2$ $[- 3yz^2 + z^3$.

12. $1728x^6 + 1728x^4y^3 + 576x^2y^6 + 64y^9$.

13. $a^3 + 3a^2b + 3a^2c + 3ab^2 + 6abc + 3ac^2 + b^3 + 3b^2c$ $[+ 3bc^2 + c^3$.

14. $8a^3 - 12a^2b + 6ab^2 - b^3$.

15. $8x^6 - 36x^5 + 114x^4 - 207x^3 + 285x^2 - 225x + 125$.

16. $27z^6 - 54az^5 + 63a^2z^4 - 44a^3z^3 + 21a^4z^2 - 6a^5z + a^6$.

17. $1 - 9y^2 + 39y^4 - 99y^6 + 156y^8 - 144y^{10} + 64y^{12}$.

18. $125x^6 - 525x^5y + 60x^4y^2 + 1547x^3y^3 - 108x^2y^4 - 1701xy^5$ $- 729y^6$.

19. $64l^{12} - 576l^{10} + 2160l^8 - 4320l^6 + 4860l^4 - 2916l^2 + 729$.

20. $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$.

21. $a^9 - 9a^8b + 36a^7b^2 - 84a^6b^3 + 126a^5b^4 - 126a^4b^5 + 84a^3b^9$ $- 36a^2b^7 + 9ab^8 - b^9$.

22. $a^3 + 6a^2b - 3a^2c + 12ab^2 - 12abc + 3ac^2 + 8b^3 - 12b^2c$ $+ 6bc^2 - c^3$.

23. $343a^6 - 441a^5b + 777a^4b^2 - 531a^3b^3 + 444a^2b^4 - 144ab^5$ $+ 64b^6$.

24. $a^{18} + 12a^{15} + 60a^{12} + 160a^9 + 240a^6 + 192a^3 + 64$.

25. $27l^{12} + 189l^{11} + 198l^{10} - 791l^9 - 594l^8 + 1701l^7 - 729l^6$.

ROOTS OF NUMBERS EXPRESSED IN ARABIC FIGURES

124. The cube root of a number expressed in Arabic figures, as in the case of square root, pp. 225–229, E. C., may be found by the process used for polynomials. An example will illustrate.

Ex. 1. Find the cube root of 389,017.

In order to decide how many digits there are in the root, we observe that $10^3 = 1000$, $100^3 = 1{,}000{,}000$. Hence the root lies between 10 and 100, that is, it contains two digits. Since $70^3 = 343{,}000$ and $80^3 = 512{,}000$, it follows that 7 is the largest number possible in tens' place. The work is arranged as follows:

$$\begin{array}{rrl}
\text{The given cube,} & 389\,017 & \underline{|70 + 3}, \text{ cube root.} \\
a^3 = 70^3 = & 343\,000 & \text{1st partial product.} \\
3a^2 = 3 \cdot 70^2 = 14700 & 46\,017 & \text{1st remainder.} \\
3ab = 3 \cdot 70 \cdot 3 = \quad 630 & & \\
b^2 = 3^2 \quad = \quad 9 & & \\
3a^2 + 3ab + b^2 = 15339 & 46\,017 & = b\,(3a^3 + 3ab + b^2). \\
 & 0 &
\end{array}$$

Having decided as above that the a of the formula is 7 tens, we cube this and subtract, obtaining 46,017 as the remaining part of the power.

The first partial divisor, $3a^2 = 14{,}700$, is divided into 46,017, giving a quotient 3, which is the b of the formula. Hence the first complete divisor, $3a^2 + 3ab + 3b^2$, is 15,339 and the product, $b(3a^2 + 3ab + b^2)$, is 46,017. Since the remainder is zero, the process ends and 73 is the cube root sought.

125. The cube of any number from 1 to 9 contains one, two, or three digits; the cube of any number between 10 and 99 contains four, five, or six digits; the cube of any number between 100 and 999 contains seven, eight, or nine digits, etc. Hence it is evident that if the digits of a number are separated into groups of three figures each, counting from units' place toward the left, the number of groups thus formed is the same as the number of digits in the root.

Ex. 2. Find the cube root of 13,997,521.

$$
\begin{array}{rrl}
\text{The given cube,} & 13\,997\,521 & \underline{|200 + 40 + 1 = 241, \text{ cube root.}} \\
a^3 = 200^3 = & 8\,000\,000 & \\
3a^2 = 120000 & 5\,997\,521 & \\
3ab = 24000 & & \\
b^2 = 1600 & & \\
\overline{145600} & \underline{5\,821\,000} = b(3a^2 + 3ab + b^2) & \\
3a'^2 = 172800 & 173\,521 & \\
3a'b' = 720 & & \\
b'^2 = 1 & & \\
\overline{173521} & \underline{173\,521} = b'(3a'^2 + 3a'b' + b'^2). & \\
 & 0 &
\end{array}
$$

Since the root contains three digits, the first one is the cube root of 8, the largest integral cube in 13.

The first partial divisor, $3 \cdot 200^2 = 120{,}000$, is completed by adding $3ab = 3 \cdot 200 \cdot 40 = 24{,}000$, and $b^2 = 1600$.

The second partial divisor, $3a'^2$, which stands for $3(200 + 40)^2 = 172{,}800$, is completed by adding $3a'b'$ which stands for $3 \cdot 240 \cdot 1 = 720$, and b'^2 which stands for 1, where a' represents the part of the root *already* found and b' the next digit to be found. At this step the remainder is zero and the root sought is 241.

EXERCISES

Find the square root of each of the following:

1. 58,081.　　2. 795,564.　　3. 11,641,744.

Find the cube root of each of the following:

4. 110,592.	7. 205,379.	10. 2,146,689.
5. 571,787.	8. 31,855,013.	11. 19,902,511.
6. 7,301,384.	9. 5,929,741.	12. 817,400,375.

126. Since the cube of a decimal fraction has three times as many places as the given decimal, it is evident that the cube root of a decimal fraction contains one decimal place for every three in the cube. Hence for the purpose of determining the places in the root, the decimal part of a cube should be divided into groups of three digits each, counting from the decimal point toward the right.

Ex. Approximate the cube root of 34.567 to two places of decimals.

$$\begin{array}{rr|l}
 & & 34.567 \;|\; \underline{3 + .2 + .05 + .007} = 3.257 \\
a^3 = 3^3 = & & 27.000 \\
3\,a^2 = 3 \cdot 3^2 = & 27. & 7.567 \\
3\,ab = 3 \cdot 3(.2) = & 1.8 & \\
b^2 = (.2)^2 = & \underline{.04} & \\
 & 28.84 & \underline{5.768} \quad = b(3\,a^2 + 3\,ab + b^2) \\
3\,a'^2 = 3(3.2)^2 = & 30.72 & 1.799000 \\
3\,a'b' = 3(3.2)(.05) = & .48 & \\
b'^2 = (.05)^2 = & \underline{.0025} & \\
 & 31.2025 & \underline{1.560125} \quad = b'(3\,a'^2 + 3\,a'b' + b'^2) \\
3\,a''^2 = 3(3.25)^2 = & 31.6875 & .238875000 \\
3a''b'' = 3(3.25)(.007) = & .06825 & \\
b''^2 = (.007)^2 = & \underline{.000049} & \\
 & 31.755799 & \underline{.222290593} = b''(3\,a''^2 + 3\,a''b'' + b''^2) \\
 & & .016584407
\end{array}$$

The decimal points are handled exactly as in arithmetic work.

127. Evidently the above process can be carried on indefinitely. 3.257 is an **approximation** to the cube root of 34.567. In fact the cube of 3.257 differs from 34.567 by less than the small fraction .017. The nearest approximation using two decimal places is 3.26. If the third decimal place were any digit less than 5, then 3.25 would be the nearest approximation using two decimal places. Hence three places must be found in order to be sure of the nearest approximation to two places.

EXERCISES

Approximate the cube root of each of the following to two places of decimals.

1. 21.4736.
2. 6.5128.
3. 58.
4. 2.
5. 3.
6. .003.
7. .3917.
8. .5.
9. .05.
10. 6410.37.
11. .001178.
12. 200.002.
13. 572.271.
14. 31.7246.
15. 51913.416.
16. Approximate the square root in Exs. 1, 2, 10, 11, and 15 of the above list.

CHAPTER VII

QUADRATIC EQUATIONS

EXPOSITION BY MEANS OF GRAPHS

128. We saw, § 65, that a single equation in two variables is satisfied by indefinitely many pairs of numbers. If such an equation is of the **first degree** in the two variables, the graph is in every case a *straight line*.

We are now to consider graphs of equations of the **second degree** in two variables. See § 66.

Ex. 1. Graph the equation $y = x^2$.

By giving various values to x and computing the corresponding values of y, we find pairs of numbers as follows which satisfy this equation:

$$\begin{cases} x = 0, \\ y = 0. \end{cases} \begin{cases} x = 1, \\ y = 1. \end{cases} \begin{cases} x = -1, \\ y = 1. \end{cases} \begin{cases} x = 2, \\ y = 4. \end{cases} \begin{cases} x = -2, \\ y = 4. \end{cases} \begin{cases} x = 3, \\ y = 9. \end{cases} \begin{cases} x = -3, \\ y = 9. \end{cases} \text{etc.}$$

These pairs of numbers correspond to points which lie on a curve as shown in Figure 3.

By referring to the graph the curve is seen to be symmetrical with respect to the y-axis. This can be seen directly from the equation itself since x is involved only as a square and hence, if $y = x^2$ is satisfied by $x = a$, $y = b$, it must also be satisfied by $x = -a$, $y = b$.

It may easily be verified that no three points of this curve lie on a straight line. The curve is called a **parabola**.

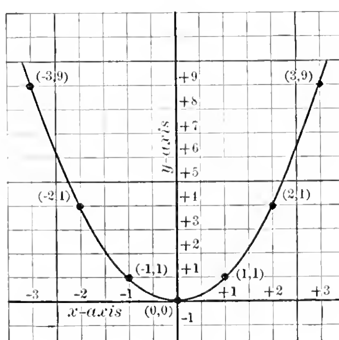

FIG. 3.

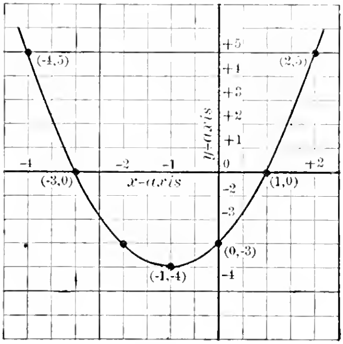

Fig. 4.

Ex. 2. Graph the equation $y = x^2 + 2x - 3$.

Each of the following pairs of numbers satisfies the equation:

$\begin{cases} x = 0, \\ y = -3. \end{cases}$ $\begin{cases} x = 1, \\ y = 0. \end{cases}$ $\begin{cases} x = -1, \\ y = -4. \end{cases}$

$\begin{cases} x = 2, \\ y = 5. \end{cases}$ $\begin{cases} x = -2, \\ y = -3. \end{cases}$ $\begin{cases} x = -3, \\ y = 0. \end{cases}$

$\begin{cases} x = -4, \\ y = 5. \end{cases}$

Plotting these points and drawing a smooth curve through them, we have the graph of the equation, as in Figure 4.

EXERCISES

In this manner graph each of the following:

1. $y = x^2 - 1.$
2. $y = x^2 + 4x.$
3. $y = x^2 + 3x - 4.$
4. $y = x^2 + 5x + 4.$
5. $y = x^2 - 7x + 6.$
6. $y = 3x^2 - 7x + 2.$
7. $y = 5x - x^2 - 4.$
8. $y = 4x - x^2 + 5.$
9. $y = x^2 + 5x - 6.$
10. $y = -x^2 + x.$
11. $y = 4x^2 - 3x - 1.$
12. $y = -4x^2 + 3x + 1.$

129. We now seek to find the points at which each of the above curves cuts the x-axis. The value of y for all points on the x-axis is zero. Hence we put $y = 0$, and try to solve the resulting equation.

Thus in Ex. 2 above, if $y = 0$, $x^2 + 2x - 3 = (x + 3)(x - 1) = 0$, which is satisfied by $x = 1$ and $x = -3$. Hence this curve cuts the x-axis in the two points $x = 1$, $y = 0$ and $x = -3$, $y = 0$, as shown in Figure 4.

Similarly in Ex. 1, if $y = 0$, $x^2 = 0$, and hence $x = 0$. Hence the curve meets the x-axis in the point $x = 0$, $y = 0$, as shown in Figure 3. On this point see § 131, Ex. 2.

EXERCISES

Find the points in which each of the twelve curves in the preceding list cuts the x-axis.

Notice that in every case the expression to the right of the equality sign can be factored, so that when $y = 0$ the resulting equation in x may be solved as in § 94.

Ex. 3. Plot the curve $y = x^2 + 4x + 2$ and find its intersection points with the x-axis.

We are not able to factor $x^2 + 4x + 2$ by inspection. Hence we solve the equation $x^2 + 4x + 2 = 0$ by completing the square as in § 175, E. C., obtaining $x = -2 + \sqrt{2}$ and $x = -2 - \sqrt{2}$. Hence the curve cuts the x-axis in points whose abscissas are given by these values of x.

In making this graph, we first plot points corresponding to *integral* values of x, as before; then, in drawing the smooth curve through these, the intersections made with the x-axis are approximately the points on the number scale corresponding to the *incommensurable* numbers, $-2 + \sqrt{2}$ and $-2 - \sqrt{2}$. See § 109.

EXERCISES

In this manner, find the points at which each of the following curves cuts the x-axis, and plot the curves. For reduction of the results to simplest forms, see §§ 159, 160, E. C.

1. $y = x^2 + 5x + 3$.
2. $y = 3x^2 + 8x - 2$.
3. $y = 6x - 4x^2 + 5$.
4. $y = -4 - 2x + 5x^2$.
5. $y = 2x - 5x^2 + 8$.
6. $y = 5 + 8x - 3x^2$.
7. $y = 3 - 9x^2 - 11x$.
8. $y = -2 - 2x + x^2$.

130. Each of the foregoing exercises involves the solution of an equation of the general form $ax^2 + bx + c = 0$. Obviously, by solving this equation, we shall obtain a formula by means of which every equation of this type may be solved. See § 179, E. C.

The two values of x are:

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

EXERCISES

By means of this formula, find the solutions of each of the following equations:

1. $2x^2 - 3x - 4 = 0.$
2. $3x^2 + 2x - 1 = 0.$
3. $3x^2 - 2x - 1 = 0.$
4. $4x^2 + 6x + 1 = 0.$
5. $x^2 - 7x + 12 = 0.$
6. $5x^2 + 8x + 3 = 0.$
7. $5x^2 - 8x + 3 = 0.$
8. $5x^2 + 8x - 3 = 0.$
9. $5x^2 - 8x - 3 = 0.$
10. $2x - 3x^2 + 7 = 0.$
11. $3x - 9x^2 + 1 = 0.$
12. $7x^2 - 3x - 2 = 0.$
13. $6x^2 + 7x + 1 = 0.$
14. $4x^2 + 5x - 3 = 0.$
15. $4x^2 - 5x - 3 = 0.$
16. $8x^2 + 3x - 5 = 0.$
17. $7x^2 + x - 3 = 0.$
18. $7x^2 - x - 4 = 0.$
19. $x^2 - 2ax = 3b - a^2.$
20. $x^2 - 6ax = 49c^2 - 9a^2.$
21. $x^2 + \dfrac{a(a+b)}{3} = ax + \dfrac{(a+b)x}{3}.$
22. $-2x^2 - \dfrac{c-d}{2}x - 2c^2x = \dfrac{c^2(c-d)}{2}.$
23. $x^2 - \dfrac{mx}{2} + 2mn = 4nx.$
24. $x^2 - 2ax + 4ab = b^2 + 3a^2.$
25. $x^2 - abx + a^2b - ax = ab^2 - bx.$
26. $x^2 + 9 - c = 6x.$
27. $nx^2 + m^2n = mn^2x + mx.$
28. $2(a+1)x^2 - (a+1)^2x + 2(a+1) = 4x.$
29. $x^2 + 9cd + 3c = (3c + 3d + 1)x.$
30. $x^2 + 2a^2 + 3a - 2 = (3a+1)x$

131. We now consider the intersections of other straight lines besides the x-axis with curves like those plotted above.

Ex. 1. Graph on the same axes the straight line, $y = -2$ and the curve, $y = x^2 + 2x - 3$.

This line is parallel to the x-axis and two units below it. It cuts the curve in the two points whose abscissas are $x_1 = -1 + \sqrt{2}$ and $x_2 = -1 - \sqrt{2}$, as found by substituting -2 for y in $y = x^2 + 2x - 3$ and solving the resulting equation in x by the formula, § 130.

Ex. 2. Graph on the same axes $y = -4$ and $y = x^2 + 2x - 3$.

This line *seems not to cut* the curve but to *touch* it at the point whose abscissa is $x = -1$.

Substituting and solving as before, we find,

$$x_1 = \frac{-2 + \sqrt{4 - 4}}{2} = \frac{-2 + 0}{2} = -1$$

and

$$x_2 = \frac{-2 - \sqrt{4 - 4}}{2} = \frac{-2 - 0}{2} = -1.$$

In this case the two values of x are *equal*, and there is only *one* point common to the line and the curve. This is understood by thinking of the line $y = -2$, in the preceding example, as moved down to the position $y = -4$, whereupon the two values of x which were *distinct* now *coincide*.

132. From the formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, it is clear that the general equation, $ax^2 + bx + c = 0$ has two *distinct* solutions unless the expression $b^2 - 4ac$ reduces to zero, in which case the two values of x *coincide*, giving $x_1 = \dfrac{-b + 0}{2a} = -\dfrac{b}{2a}$ and $x_2 = \dfrac{-b - 0}{2a} = -\dfrac{b}{2a}$.

Ex. 1. In $2x^2 - 9x + 8 = 0$, determine without solving whether the two values of x are distinct or coincident.

In this case, $a = 2,\ b = -9,\ c = 8.$

Hence $b^2 - 4ac = 81 - 64 = 17.$

Hence the values of x are distinct.

Ex. 2. In $4x^2 - 12x + 9 = 0$, determine whether the values of x are distinct or coincident.

In this case, $b^2 - 4ac = 144 - 4 \cdot 4 \cdot 9 = 0$. Hence the values of x are *coincident*.

EXERCISES

In each of the following, determine without solving whether the two solutions are distinct or coincident:

1. $x^2 - 7x + 4 = 0$.
2. $4x^2 + 28x + 49 = 0$.
3. $9x^2 + 12x + 4 = 0$.
4. $x^2 + 6x + 9 = 0$.
5. $-x^2 + 9x + 25 = 0$.
6. $6x^2 - 3x - 1 = 0$.
7. $4x^2 - 16x + 16 = 0$.
8. $8x^2 - 13 = 4x$.
9. $12x^2 - 18 = 24x$.
10. $16x^2 - 56x = -49$.

133. Definition. A line which cuts a curve in two *coincident points* is said to be **tangent to the curve.**

134. Problem. What is the value of a in $y = a$, if this line is tangent to the curve $y = x^2 + 5x + 8$?

Substituting a for y and solving by means of the formula, we have

$$x = \frac{-5 \pm \sqrt{25 - 4(8 - a)}}{2}.$$

If the line is to be tangent to the curve, then the expression under the radical sign must be zero so that the two values of x may coincide. That is, $25 - 4(8 - a) = 0$, or $a = \frac{7}{4}$.

On plotting the curve, the line $y = \frac{7}{4}$ is found to be tangent to it.

EXERCISES

In the first 18 exercises on p. 86 obtain equations of curves by letting the left members equal y. Then find the equations of straight lines, $y = a$, which are tangent to these curves.

135. Problem. Find the intersection points of the curve $y = x^2 + 3x + 5$ and the line $y = 2\frac{1}{2}$.

Substituting for y and solving for x we have

$$x_1 = \frac{-6 + \sqrt{36 - 40}}{4} = \frac{-6 + 2\sqrt{-1}}{4} = \frac{-3 + \sqrt{-1}}{2};$$

$$x_2 = \frac{-6 - \sqrt{36 - 40}}{4} = \frac{-6 - 2\sqrt{-1}}{4} = \frac{-3 - \sqrt{-1}}{2}.$$

These results involve the **imaginary unit** already noticed in § 112. Numbers of the type $a + b\sqrt{-1}$ are discussed further in § 195. For the present we will regard such results as merely indicating that the conditions stated by the equations cannot be fulfilled by *real numbers*. This means that the curve and the line have *no point in common*, as is evident on constructing the graphs.

By proceeding as in § 134 we find that the line $y = \frac{11}{4}$ is *tangent* to the curve $y = x^2 + 3x + 5$. Clearly all lines $y = a$, in which $a > \frac{11}{4}$, are *above* this line and hence cut this curve in *two points*.

All such lines for which $a < \frac{11}{4}$ are *below* the line $y = \frac{11}{4}$ and hence do *not meet* the curve at all.

Solving $y = a$ and $y = x^2 + 3x + 5$ for x by first substituting a for y we have

$$x = \frac{-3 \pm \sqrt{4a - 11}}{2}.$$

If $a > \frac{11}{4}$ the number under the radical sign is *positive*, and there are *two real and distinct* values of x. Hence the line and the curve meet in two points.

If $a < \frac{11}{4}$, the number under the radical sign is *negative*. Consequently the values of x are *imaginary* and the line and the curve do not meet.

Hence we see that the conclusions obtained from the solution of the equations agree with those obtained from the graphs.

136. From the two preceding problems it appears that it is possible to determine the *relative* positions of the line and the curve *without completely solving* the equations. Namely, as soon as y is eliminated and the equation in x is reduced to the form $ax^2 + bx + c = 0$, we examine $b^2 - 4ac$ as follows:

(1) If $b^2 - 4ac > 0$, *i.e. positive*, then the line cuts the curve in two distinct points.

(2) If $b^2 - 4ac = 0$, then the line is tangent to the curve. See § 133.

(3) If $b^2 - 4ac < 0$, *i.e. negative*, then the line does not cut the curve.

137. Problem. Find the points of intersection of

$$y = x^2 + 3x + 13 \quad (1), \quad \text{and} \quad y + 3x = 7 \quad (2).$$

Eliminating y and reducing the resulting equation in x to the form $ax^2 + bx + c = 0$, we have $x^2 + 6x + 6 = 0$.

Solving, $\qquad x_1 = -3 + \sqrt{3},\ x_2 = -3 - \sqrt{3}.$

Substituting these values of x in (2) and solving for y, we have

$$\begin{cases} x_1 = -3 + \sqrt{3} \\ y_1 = 16 - 3\sqrt{3} \end{cases} \quad \text{and} \quad \begin{cases} x_2 = -3 - \sqrt{3} \\ y_2 = 16 + 3\sqrt{3} \end{cases}$$

which are the points in which the line meets the curve.

Here $b^2 - 4ac = 12$, which shows in advance that there are *two points* of intersection.

EXERCISES

In each of the following determine without graphing whether or not the line meets the curve, and in case it does, find the intersection points:

1. $\begin{cases} y = 2x^2 - 3x - 4, \\ y - x = 3. \end{cases}$

2. $\begin{cases} y = 2x^2 + 2x - 1, \\ 2y = x - 1. \end{cases}$

3. $\begin{cases} y = 3x^2 - 2x - 1, \\ 2x - y = 4. \end{cases}$

4. $\begin{cases} y = 4x^2 + 6x + 1, \\ x = y + 5. \end{cases}$

5. $\begin{cases} y = x^2 - 7x + 12, \\ 5x - y = -1. \end{cases}$

6. $\begin{cases} y = 5x^2 + 8x + 3, \\ 2y - 5x - 2 = 0. \end{cases}$

7. $\begin{cases} y = 5x^2 - 8x + 3, \\ 3 - x = 3y. \end{cases}$

8. $\begin{cases} y = -5x^2 + 8x - 3, \\ 2 - 4y - x = 0. \end{cases}$

9. $\begin{cases} y = -5x^2 - 8x - 3, \\ 5y - 3x = 8. \end{cases}$

10. $\begin{cases} y = 3x - 3x^2 + 7, \\ -5 - 3x + 2y = 0. \end{cases}$

138. Problem. Graph the equation $x^2 + y^2 = 25$.

Writing the equation in the form $y = \pm\sqrt{25 - x^2}$, and assigning values to x, we compute the corresponding values of y as follows:

$$\begin{cases} x = 0, \\ y = \pm 5. \end{cases} \quad \begin{cases} x = \pm 5, \\ y = 0. \end{cases} \quad \begin{cases} x = 3, \\ y = \pm 4. \end{cases} \quad \begin{cases} x = -3, \\ y = \pm 4. \end{cases} \quad \begin{cases} x = 4, \\ y = \pm 3. \end{cases} \quad \begin{cases} x = -4, \\ y = \pm 3. \end{cases}$$

Evidently, for x greater than 5 in absolute value, the corresponding y's are *imaginary*, and for each x between -5 and $+5$ there are two y's equal in absolute value, but with opposite signs.

It seems apparent that these points lie on the circumference of a circle whose radius is 5, as shown in Figure 5. Indeed, if we consider any point x_1, y_1 on this circumference, it is evident that $x_1^2 + y_1^2 = 25$, since the sum of the squares on the sides of a right triangle is equal to the square on the hypotenuse. (See figure, p. 207, E. C.)

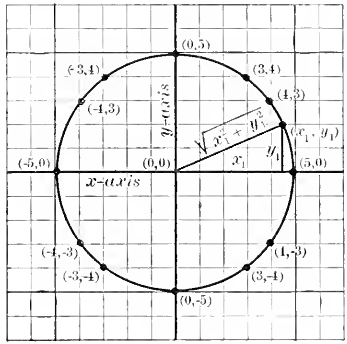

FIG. 5.

The equation $x^2 + y^2 = 25$ is, therefore, the equation of a circle with radius 5. Similarly, $x^2 + y^2 = r^2$ is the equation of a circle with center at the point (0, 0) and radius r.

139. Problem. Find the points of intersection of the circle $x^2 + y^2 = 25$ and the line $x + y = 7$.

Eliminating y from these equations, and reducing the equation in x to the form $ax^2 + bx + c = 0$, we have

$$x^2 - 7x + 12 = 0.$$

From which $$x_1 = 4, \quad x_2 = 3.$$

Substituting these values of x in $x + y = 7$, we have $y_1 = 3$, $y_2 = 4$. Hence $x_1 = 4$, $y_1 = 3$ and $x_2 = 3$, $y_2 = 4$ are the required points.

Verify this by graphing the two equations on the same axes.

140. Problem. Find the points of intersection of the circle $x^2 + y^2 = 25$ and the line $3x + 4y = 25$.

Eliminating y and solving for x, we find $x = \dfrac{6 \pm 0}{2} = 3$.

Hence $x_1 = x_2 = 3$, from which $y_1 = y_2 = 4$.

Since the two values of x coincide, and likewise the two values of y, the circumference and the line have but *one point* in common. Verify by graphing the line and the circle on the same axes.

141. Problem. Find the points of intersection of

$$x^2 + y^2 = 25$$
$$\text{and } x + y = 10.$$

Substituting for y and solving for x we have

$$x = \frac{20 \pm \sqrt{400 - 600}}{4} = \frac{20 \pm \sqrt{-200}}{4}$$
$$= \frac{20 \pm 10\sqrt{-2}}{4} = \frac{10 \pm 5\sqrt{-2}}{2}.$$

The imaginary values of x indicate that there is no intersection point. Verify by plotting.

EXERCISES

In each of the following determine by solving whether the line and the circumference meet, and in case they do, find the points of intersection. Verify each by constructing the graph.

1. $\begin{cases} x^2 + y^2 = 16, \\ x + y = 4. \end{cases}$

2. $\begin{cases} x^2 + y^2 = 36, \\ 4x + y = 6. \end{cases}$

3. $\begin{cases} x^2 + y^2 = 25, \\ 2x + y = -5. \end{cases}$

4. $\begin{cases} x^2 + y^2 = 20, \\ 2x + y = 0. \end{cases}$

5. $\begin{cases} x^2 + y^2 = 7, \\ x + y = 8. \end{cases}$

6. $\begin{cases} x^2 + y^2 = 8, \\ x - y = 4. \end{cases}$

7. $\begin{cases} x^2 + y^2 = 41, \\ x - 3y = 7. \end{cases}$

8. $\begin{cases} x^2 + y^2 = 29, \\ 3x - 7y = -29. \end{cases}$

9. $\begin{cases} x^2 + y^2 = 12, \\ x - y = 6. \end{cases}$

10. $\begin{cases} x^2 + y^2 = 4, \\ 2x - 3y = 4. \end{cases}$

11. $\begin{cases} x^2 + y^2 = 40, \\ x + 2y = 10. \end{cases}$

12. $\begin{cases} x^2 + y^2 = 25, \\ x + y = 9. \end{cases}$

142. Problem. Graph on the same axes the circle, $x^2 + y^2 = 5^2$, and the lines, $3x + 4y = 20$, $3x + 4y = 25$, and $3x + 4y = 30$.

The first line *cuts* the circumference in two distinct points, the second seems to be *tangent* to it, and the third does *not meet* it. Observe that the three lines are parallel. See Figure 6.

In order to discuss the relative positions of such straight lines and the circumference of a circle, we solve the following equations simultaneously:

$$x^2 + y^2 = r^2 \qquad (1)$$

$$3x + 4y = c \qquad (2)$$

Eliminating y by substitution, and solving for x, we find

$$x = \frac{3c \pm 4\sqrt{25r^2 - c^2}}{25}. \qquad (3)$$

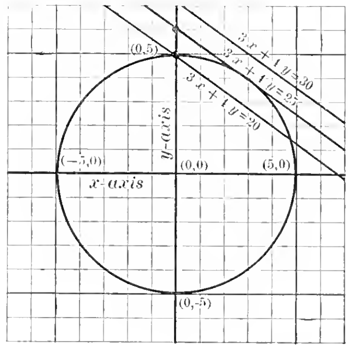

FIG. 6.

The two values of x from (3) are the abscissas of the points of intersection of the circumference (1) and the line (2).

These values of x are *real and distinct* if $25r^2 - c^2$ is *positive*, *real and coincident* if $25r^2 - c^2 = 0$, and *imaginary* if $25r^2 - c^2$ is *negative*.

Now $25r^2 - c^2$ is *positive* if $r = 5$, $c = 20$; *zero* if $r = 5$, $c = 25$; and *negative* if $r = 5$, $c = 30$.

Hence these results obtained from the solution of the equations agree with the facts observed in the graphs above.

143. Definition. Letters such as c and r in the above solution to which any arbitrary constant values may be assigned are called **parameters**, while x and y are the **unknowns** of the equations.

EXERCISES

Solve each of the following pairs of equations.

Give such values to the parameters involved that the line (a) may cut the curve in two distinct points, (b) may be tangent to the curve, (c) shall fail to meet the curve.

1. $\begin{cases} x^2 + y^2 = 4, \\ ax + 3y = 16. \end{cases}$

2. $\begin{cases} x^2 + y^2 = 16, \\ 2x + by = 12. \end{cases}$

3. $\begin{cases} x^2 + y^2 = 25, \\ 2x + 3y = c. \end{cases}$

4. $\begin{cases} y^2 = 8x, \\ 3x + 4y = c. \end{cases}$

5. $\begin{cases} 5y^2 = 2px, \\ x + y = c. \end{cases}$

9. $\begin{cases} y = 3x^2 + mx - m, \\ 2x + 2y + 1 = 0. \end{cases}$

6. $\begin{cases} y = x^2 + mx + m, \\ x + y = 4. \end{cases}$

10. $\begin{cases} y = mx^2 + 2nx, \\ 2y - bx - 5 = 0. \end{cases}$

7. $\begin{cases} y = mx^2 - nx - 4, \\ x - 3y = 8. \end{cases}$

11. $\begin{cases} y = x^2 + nx + 1, \\ ax + 2y = 10. \end{cases}$

8. $\begin{cases} y = 2x^2 - 3x + 1, \\ 2x - by - 1 = 0. \end{cases}$

12. $\begin{cases} x^2 + y^2 = r^2, \\ ax + by = c. \end{cases}$

144. Problem. Graph the equation $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

Writing the equation in the form $y = \pm\frac{4}{5}\sqrt{25 - x^2}$, and assigning values to x, we compute the corresponding values of y as follows:

$\begin{cases} x = 0, \\ y = \pm 4, \end{cases}$ $\begin{cases} x = \pm 5, \\ y = 0, \end{cases}$ $\begin{cases} x = 1, \\ y = \pm 3.9, \end{cases}$ $\begin{cases} x = -1, \\ y = \pm 3.9, \end{cases}$

$\begin{cases} x = 2, \\ y = \pm 3.7, \end{cases}$ $\begin{cases} x = 3, \\ y = \pm 3.2, \end{cases}$ $\begin{cases} x = 4, \\ y = \pm 2.4. \end{cases}$

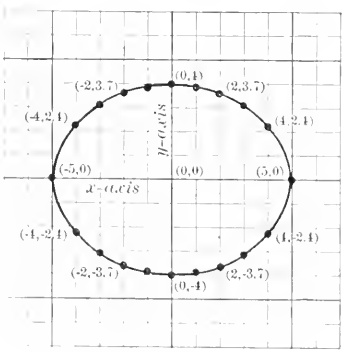

FIG. 7.

Evidently if x is greater than 5 in absolute value, the corresponding values of y are imaginary.

Plotting these points, they are found to lie on the curve shown in Figure 7. This curve is called an **ellipse**.

EXERCISES

Solve the following pairs of equations.

In this way determine whether the straight line and the curve intersect, and in case they do, determine the coördinates of the intersection points. Verify each by constructing the graphs.

1. $\begin{cases} \frac{x^2}{16} + \frac{y^2}{9} = 1, \\ 3x + 4y = 12. \end{cases}$

2. $\begin{cases} \frac{x^2}{49} + \frac{y^2}{16} = 1, \\ 2x - 7y = 8. \end{cases}$

3. $\begin{cases} x^2 + 4y^2 = 25, \\ 2x - y = 4. \end{cases}$

4. $\begin{cases} 3x^2 + 2y^2 = 11, \\ x - 3y = 7. \end{cases}$

5. $\begin{cases} \dfrac{x^2}{25} + \dfrac{y^2}{9} = 1, \\ 2x - y = 14. \end{cases}$

6. $\begin{cases} y = 2x^2 - 3x + 4, \\ y - 4x - 8 = 0. \end{cases}$

7. $\begin{cases} x^2 + y^2 = 16, \\ x + y = 7. \end{cases}$

8. $\begin{cases} \dfrac{x^2}{64} + \dfrac{y^2}{12} = 1, \\ 4y - 2x = 4. \end{cases}$

9. $\begin{cases} \dfrac{x^2}{36} + \dfrac{y^2}{45} = 1, \\ -5x + 6y = 10. \end{cases}$

10. $\begin{cases} \dfrac{x^2}{49} + \dfrac{y^2}{25} = 1, \\ x + y = 12. \end{cases}$

When arbitrary constants are introduced in the equations of a straight line and an ellipse, we may determine values for these constants so as to make the line cut the ellipse, touch it, or not cut it, as in the case of the circle, § 142.

EXERCISES

Solve each of the following pairs simultaneously.

Give such values to the constants that the line shall (*a*) cut the curve in two distinct points, (*b*) be a tangent to the curve, (*c*) have no point in common with the curve.

In case (*b*) is found very difficult, this may be omitted.

1. $\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{16} = 1, \\ 8x + 5y = 40. \end{cases}$

2. $\begin{cases} \dfrac{x^2}{25} + \dfrac{y^2}{b^2} = 1, \\ 4x + 15y = 60. \end{cases}$

3. $\begin{cases} \dfrac{x^2}{25} + \dfrac{y^2}{16} = 1, \\ 4x - 5y = c. \end{cases}$

4. $\begin{cases} \dfrac{x^2}{16} + \dfrac{y^2}{25} = 1, \\ 5x - by = 20. \end{cases}$

5. $\begin{cases} \dfrac{x^2}{16} + \dfrac{y^2}{25} = 1, \\ ax + 4y = 20. \end{cases}$

6. $\begin{cases} \dfrac{x^2}{36} + \dfrac{y^2}{16} = 1, \\ ax + 6y - 60 = 0. \end{cases}$

7. $\begin{cases} \dfrac{x^2}{36} + \dfrac{y^2}{25} = 1, \\ 5x + by = 60. \end{cases}$

8. $\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, \\ bx - 2y = 5. \end{cases}$

9. $\begin{cases} x^2 + y^2 = r^2, \\ ax - 3y = 4. \end{cases}$

10. $\begin{cases} 5x^2 + 3y^2 = 16, \\ hx - ky = 8. \end{cases}$

11. $\begin{cases} x^2 + 7y^2 = 144, \\ ax + by = 12. \end{cases}$

12. $\begin{cases} x^2 + 4y^2 = r^2, \\ ax + by = c. \end{cases}$

SPECIAL METHODS OF SOLUTION

145. We have thus far solved simultaneously one equation of the second degree with one of the first degree. After substitution each has reduced to the solution of an ordinary quadratic, namely, of the form, $ax^2 + bx + c = 0$.

While this is an effective general method, yet some important special forms of solution are shown in the following examples:

Ex. 1. Solve $\begin{cases} x^2 + y^2 = a, & (1) \\ x - y = b. & (2) \end{cases}$

Square both members of (2) and subtract from (1).

$$2\,xy = a - b^2. \qquad (3)$$

Add (1) and (3). $\quad x^2 + 2\,xy + y^2 = 2\,a - b^2. \qquad (4)$

Hence $\quad x + y = \pm\sqrt{2\,a - b^2}. \qquad (5)$

From (2) and (5), adding and subtracting

$$\begin{cases} x_1 = \dfrac{\sqrt{2\,a - b^2} + b}{2}, \\[2ex] y_1 = \dfrac{\sqrt{2\,a - b^2} - b}{2}, \end{cases} \text{and} \quad \begin{cases} x_2 = \dfrac{-\sqrt{2\,a - b^2} + b}{2}, \\[2ex] y^2 = \dfrac{-\sqrt{2\,a - b^2} - b}{2}. \end{cases}$$

Ex. 2. Solve $\begin{cases} x^2 - y^2 = a, & (1) \\ x - y = b. & (2) \end{cases}$

From (1) $\quad (x - y)(x + y) = a. \qquad (3)$

Substituting b for $x - y$ in (3), $\quad x + y = \dfrac{a}{b}. \qquad (4)$

Then (2) and (4) may be solved as above.

Ex. 3. Solve $\begin{cases} x + y = a, & (1) \\ xy = b. & (2) \end{cases}$

Multiply (2) by 4, subtract from the square of (1), and get

$$x^2 - 2\,xy + y^2 = a^2 - 4\,b, \qquad (3)$$

whence, $\quad x - y = \pm\sqrt{a^2 - 4\,b}. \qquad (4)$

Then (1) and (4) may be solved as in Ex. 1.

The equations $\begin{cases} x - y = a, \\ xy = b, \end{cases}$

may be solved in a similar manner.

146. We are now to study the solution of a pair of equations each of the second degree. See § 66.

Consider
$$x^2 + y = a, \tag{1}$$
$$x + y^2 = b. \tag{2}$$

Solving (1) for y and substituting in (2) we have,

$$x + a^2 - 2\,ax^2 + x^4 = b,$$

which is of the fourth degree and cannot be solved by any methods thus far studied. There are, however, special cases in which two equations each of the second degree can be solved by a proper combination of methods already known.

147. Case I. *When only the squares of the unknowns enter the equations.*

Example. Solve $\begin{cases} a_1x^2 + b_1y^2 = c_1, \\ a_2x^2 + b_2y^2 = c_2. \end{cases}$

These equations are *linear* if x^2 and y^2 are regarded as the unknowns.

Solving for x^2 and y^2 as in § 73, we obtain,

$$x^2 = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}, \quad y^2 = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

Hence, taking square roots,

$$\begin{cases} x_1 = \sqrt{\dfrac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}}, \\ y_1 = \sqrt{\dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}}, \end{cases} \qquad \begin{cases} x_3 = \sqrt{\dfrac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}}, \\ y_3 = -\sqrt{\dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}}, \end{cases}$$

$$\begin{cases} x_2 = -\sqrt{\dfrac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}}, \\ y_2 = \sqrt{\dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}}, \end{cases} \qquad \begin{cases} x_4 = -\sqrt{\dfrac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}}, \\ y_4 = -\sqrt{\dfrac{a_1c_2 - a_2a_1}{a_1b_2 - a_2b_1}}. \end{cases}$$

In this case there are four pairs of numbers which satisfy the two equations. This is in general true of two equations each of the second degree.

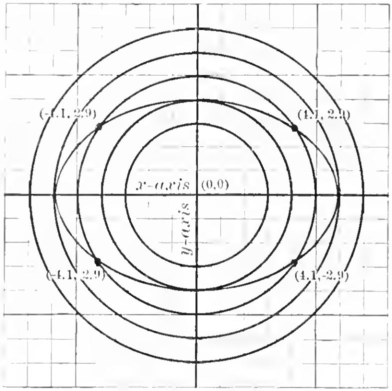

FIG. 8.

Example. Solve simultaneously, obtaining results to one decimal place:

$$\begin{cases} \dfrac{x^2}{36} + \dfrac{y^2}{16} = 1, & (1) \\ x^2 + y^2 = 25. & (2) \end{cases}$$

Clear (1) of fractions and proceed as above. Verify the solution by reference to the graph given in Figure 8.

EXERCISES

Solve simultaneously each of the following pairs of equations and interpret all the solutions in each case from the graph in Figure 8:

1. $\begin{cases} \dfrac{x^2}{36} + \dfrac{y^2}{16} = 1, \\ x^2 + y^2 = 36. \end{cases}$

2. $\begin{cases} \dfrac{x^2}{36} + \dfrac{y^2}{16} = 1, \\ x^2 + y^2 = 16. \end{cases}$

3. $\begin{cases} \dfrac{x^2}{36} + \dfrac{y^2}{16} = 1, \\ x^2 + y^2 = 49. \end{cases}$

4. $\begin{cases} \dfrac{x^2}{36} + \dfrac{y^2}{16} = 1, \\ x^2 + y^2 = 9. \end{cases}$

148. Problem. Graph the equation $\dfrac{x^2}{25} - \dfrac{y^2}{16} = 1.$

Writing the equation in the form $y = \pm \frac{4}{5}\sqrt{x^2 - 25}$, and assigning values to x, we compute the corresponding values of y exactly or approximately as follows:

$\begin{cases} x = \pm 5, \\ y = 0, \end{cases}$ $\begin{cases} x = 6\frac{1}{4}, \\ y = \pm 3, \end{cases}$ $\begin{cases} x = -6\frac{1}{4}, \\ y = \pm 3, \end{cases}$ $\begin{cases} x = 7, \\ y = \pm 3.9, \end{cases}$ $\begin{cases} x = -7, \\ y = \pm 3.9, \end{cases}$ $\begin{cases} x = 8, \\ y = \pm 5, \end{cases}$ $\begin{cases} x = -8, \\ y = \pm 5. \end{cases}$

Evidently when x is less than 5 in absolute value, y is imaginary, and as x increases beyond 8 in absolute value, y continually increases.

Plotting these points, they are found to lie on the curve as shown in Figure 9. This curve is called a **hyperbola**.

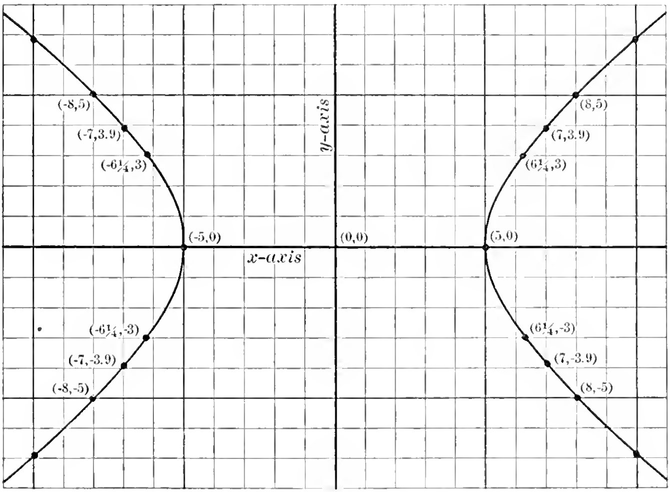

FIG. 9.

EXERCISES

Solve each of the following pairs of equations.

Construct a graph similar to the one in Figure 8 which shall contain the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ and the circles given in Exs. 1, 2, and 3. Construct another graph containing the same hyperbola and the ellipses given in Exs. 4, 5, and 6. From these graphs interpret the solutions of each pair of equations.

1. $\begin{cases} \dfrac{x^2}{25} - \dfrac{y^2}{16} = 1, \\ x^2 + y^2 = 16. \end{cases}$ **2.** $\begin{cases} \dfrac{x^2}{25} - \dfrac{y^2}{16} = 1, \\ x^2 + y^2 = 25. \end{cases}$ **3.** $\begin{cases} \dfrac{x^2}{25} - \dfrac{y^2}{16} = 1, \\ x^2 + y^2 = 36. \end{cases}$

4. $\begin{cases} \dfrac{x^2}{25} - \dfrac{y^2}{16} = 1, \\ \dfrac{x^2}{36} + \dfrac{y^2}{16} = 1. \end{cases}$ 5. $\begin{cases} \dfrac{x^2}{25} - \dfrac{y^2}{16} = 1, \\ \dfrac{x^2}{25} + \dfrac{y^2}{16} = 1. \end{cases}$ 6. $\begin{cases} \dfrac{x^2}{25} - \dfrac{y^2}{16} = 1, \\ \dfrac{x^2}{16} + \dfrac{y^2}{9} = 1. \end{cases}$

7. Graph the equation $xy = 9$.

Graph $xy = 8$ on the same axes with each of the following:

8. $x^2 + y^2 = 16$. 9. $x^2 + y^2 = 25$. 10. $x^2 + y^2 = 4$.

11. $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$. 12. $\dfrac{x^2}{25} + \dfrac{y^2}{10.24} = 1$. 13. $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$.

14. $\dfrac{x^2}{25} - \dfrac{y^2}{16} = 1$. 15. $\dfrac{x^2}{25} - \dfrac{y^2}{9} = 1$. 16. $\dfrac{x^2}{16} - \dfrac{y^2}{4} = 1$.

17. From those graphs in Exs. 8 to 16, in which the curves meet, determine as accurately as possible by measurement the coördinates of the points of intersection or tangency.

18. Solve simultaneously the pairs of equations given in Exs. 8 to 10, after studying the method explained in § 150, Ex. 1. Compare the results with those obtained from the graphs.

19. Solve Exs. 11 to 16 by the method explained in § 149, and compare the results with those obtained from the graphs.

149. Case II. *When all terms containing the unknowns are of the second degree in the unknowns.*

Example. Solve
$$\begin{cases} 2x^2 - 3xy + 4y^2 = 3, & (1) \\ 3x^2 - 4xy + 3y^2 = 2. & (2) \end{cases}$$

Put $y = vx$ in (1) and (2), obtaining

$$\begin{cases} x^2(2 - 3v + 4v^2) = 3, & (3) \\ x^2(3 - 4v + 3v^2) = 2. & (4) \end{cases}$$

Hence from (3) and (4),

$$x^2 = \frac{3}{2 - 3v + 4v^2}, \text{ and also } x^2 = \frac{2}{3 - 4v + 3v^2}. \qquad (5)$$

From (5)
$$\frac{3}{2-3v+4v^2}=\frac{2}{3-4v+3v^2}, \qquad (6)$$

or
$$v^2-6v+5=0. \qquad (7)$$

Hence
$$v=1, \text{ and } v=5. \qquad (8)$$

From $y=vx$,
$$y=x, \text{ and } y=5x. \qquad (9)$$

If $y=x$, then from (1) and (2),

$$\begin{cases} x=1, \\ y=1, \end{cases} \text{ and } \begin{cases} x=-1, \\ y=-1. \end{cases}$$

If $y=5x$, then from (1) and (2),

$$\begin{cases} x=\dfrac{1}{\sqrt{29}}, \\ y=\dfrac{5}{\sqrt{29}}, \end{cases} \text{ and } \begin{cases} x=-\dfrac{1}{\sqrt{29}}, \\ y=-\dfrac{5}{\sqrt{29}}. \end{cases}$$

Verify each of these four solutions by substituting in equations (1) and (2).

150. There are many other special forms of simultaneous equations which can be solved by proper combination of the methods thus far used. Also, many pairs of equations of a degree higher than the second in the two unknowns may be solved by means of quadratic equations.

The suggestions given in the following examples illustrate the devices in most common use.

The solution should in each case be completed by the student.

Ex. 1. Solve
$$\begin{cases} x^2+y^2=58, & (1) \\ xy=21. & (2) \end{cases}$$

Adding twice (2) to (1) and taking square roots, we have

$$x+y=10, \text{ and } x+y=-10. \qquad (3)$$

Each of the equations (3) may now be solved simultaneously with (2), as in Ex. 3, p. 96.

Ex. 2. Solve
$$\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=5, & (1) \\ \dfrac{1}{x^2}+\dfrac{1}{y^2}=13. & (2) \end{cases}$$

Let $\dfrac{1}{x}=a$ and $\dfrac{1}{y}=b$. Then these equations reduce to
$$\begin{cases} a+b=5, & (3) \\ a^2+b^2=13. & (4) \end{cases}$$

(3) and (4) may then be solved as in Ex. 1, p. 96.

Ex. 3. Solve
$$\begin{cases} x^2+y^2+x+y=8, & (1) \\ xy=2. & (2) \end{cases}$$

Add twice (2) to (1), obtaining
$$x^2+2xy+y^2+x+y=12. \qquad (3)$$

Let $x+y=a$. Then (3) reduces to
$$a^2+a=12,$$
or,
$$a=3,\ a=-4. \qquad (4)$$

Hence $\qquad x+y=3,$ and $x+y=-4. \qquad (5)$

Now solve each equation in (5) simultaneously with (2).

Ex. 4. Solve
$$\begin{cases} x^4y^4+x^2y^2=272, & (1) \\ x^2+y^2=10. & (2) \end{cases}$$

In (1) substitute a for x^2y^2. Then
$$a^2+a=272, \text{ whence } a=16, \text{ and } -17.$$

Hence $\qquad xy=\pm\sqrt{16}=\pm 4,$ and $\pm\sqrt{-17}.$

Each of these equations may now be solved simultaneously with (2), as in Ex. 1, p. 101.

Ex. 5. Solve
$$\begin{cases} x^3-y^3=117, & (1) \\ x-y=3. & (2) \end{cases}$$

By factoring, (1) becomes

$$(x - y)(x^2 + xy + y^2) = 117. \tag{3}$$

Substituting 3 for $x - y$, we have

$$x^2 + xy + y^2 = 39. \tag{4}$$

(2) and (4) may now be solved by substitution as in §§ 140–144.

Ex. 6. Solve
$$\begin{cases} x^3 + y^3 = 513, & (1) \\ x + y = 9. & (2) \end{cases}$$

Factor (1) and substitute 9 for $x + y$. Then proceed as in Ex. 5.

Ex. 7. Solve
$$\begin{cases} x^2y + xy^2 = 126, & (1) \\ x + y = 9. & (2) \end{cases}$$

Factoring (1) and substituting 9 for $x + y$, we have

$$xy = 14. \tag{3}$$

(2) and (3) may then be solved as in Ex. 3, p. 96.

Ex. 8. Solve
$$\begin{cases} x^3 + y^3 = 54\,xy, & (1) \\ x + y = 6. & (2) \end{cases}$$

Factor (1) and substitute 6 for $x + y$, obtaining

$$x^2 - xy + y^2 = 9\,xy. \tag{3}$$

(2) and (3) may now be solved by substitution, as in §§ 140–144.

Ex. 9. Solve
$$\begin{cases} x^3 - y^3 = 63, & (1) \\ x^2 + xy + y^2 = 21. & (2) \end{cases}$$

Factor (1) and substitute 21 for $x^2 + xy + y^2$, then proceed as in Ex. 8.

Ex. 10. Solve
$$\begin{cases} x^3 + y^3 = 243, & (1) \\ x^2y + xy^2 = 162. & (2) \end{cases}$$

Multiply (2) by 3 and add to (1), obtaining a perfect cube. Taking cube roots, we have

$$x + y = 9. \tag{3}$$

(1) and (3) are now solved as in the preceding example.

Ex. 11. Solve
$$\begin{cases} x^4 + y^4 = 641, & (1)\\ x + y = 7. & (2)\end{cases}$$

Raise (2) to the fourth power and subtract (1), obtaining

$$4\,x^3y + 6x^2y^2 + 4\,xy^3 = 1760. \qquad (3)$$

Factoring, $$2\,xy\,(2\,x^2 + 3\,xy + 2\,y^2) = 1760. \qquad (4)$$

Squaring (2) we have

$$2\,x^2 + 4\,xy + 2\,y^2 = 98, \qquad (5)$$

or
$$2\,x^2 + 3\,xy + 2\,y^2 = 98 - xy. \qquad (6)$$

Substituting (6) in (4), we have

$$2\,xy\,(98 - xy) = 1760, \qquad (7)$$

or
$$x^2y^2 - 98\,xy + 880 = 0. \qquad (8)$$

In (8) put $xy = a$, obtaining

$$a^2 - 98\,a + 880 = 0. \qquad (9)$$

The solution of (9) gives two values for xy, each of which may now be combined with (2) as in Ex. 3, p. 96.

EXERCISES

Solve each of the following pairs of equations:

1. $\begin{cases} r^2 + rs + s^2 = 63,\\ r - s = 3.\end{cases}$

2. $\begin{cases} 3\,x^2 + 2\,y^2 = 35,\\ 2\,x^2 - 3\,y^2 = 6.\end{cases}$

3. $\begin{cases} 3\,x^2 + 2\,xy = 16,\\ 4\,x^2 - 3\,xy = 10.\end{cases}$

4. $\begin{cases} a^2 + ab + b^2 = 7,\\ a^2 - ab + b^2 = 19.\end{cases}$

5. $\begin{cases} x^2 + y^2 = a,\\ xy = b.\end{cases}$

6. $\begin{cases} x^2 + y^2 = a,\\ x^2 + z^2 = b,\\ y^2 + z^2 = c.\end{cases}$

7. $\begin{cases} ax - by = 0,\\ x^2 + y^2 = c.\end{cases}$

8. $\begin{cases} x^2 + xy = a,\\ y^2 + xy = b.\end{cases}$

9. $\begin{cases} x^3 + y^3 = 91,\\ x + y = 7.\end{cases}$

10. $\begin{cases} x^2 + y^2 = a,\\ x^2 - y^2 = b.\end{cases}$

11. $\begin{cases} x^2 - 3\,xy = 0,\\ 5\,x^2 + 3\,y^2 = 9.\end{cases}$

12. $\begin{cases} \dfrac{1}{x^3} + \dfrac{1}{y^3} = 19,\\[2mm] \dfrac{1}{x} + \dfrac{1}{y} = 1.\end{cases}$

13. $\begin{cases} 3\,x - 2\,y = 6,\\ 3\,x^2 - 2\,xy + 4\,y^2 = 12.\end{cases}$

14. $\begin{cases} a + b + ab = 11,\\ (a + b)^2 + a^2b^2 = 61.\end{cases}$

15. $\begin{cases} \dfrac{1}{a^2} + \dfrac{1}{ab} + \dfrac{1}{b^2} = 19,\\[2mm] \dfrac{1}{a} + \dfrac{1}{b} = 8.\end{cases}$

16. $\begin{cases} 4a^2 - 2ab = b^2 - 16, \\ 5a^2 = 7ab - 36. \end{cases}$

17. $\begin{cases} 3x^2 - 9y^2 = 12, \\ 2x - 3y = 14. \end{cases}$

18. $\begin{cases} x^2 + xy + y^2 = a, \\ x^2 + y^2 = b. \end{cases}$

19. $\begin{cases} x^2 + y^2 + x + y = 18, \\ xy = 6. \end{cases}$

20. $\begin{cases} x^2 + y^2 + x - y = 36, \\ xy = 15. \end{cases}$

21. $\begin{cases} x^2 - 5xy + y^2 = -2, \\ x^2 + 7xy + y^2 = 22. \end{cases}$

22. $\begin{cases} a^2 + 6ab + b^2 = 124, \\ a + b = 8. \end{cases}$

23. $\begin{cases} a^2 - 3ab + 2b^2 = 0, \\ 2a^2 + ab - b^2 = 9. \end{cases}$

24. $\begin{cases} x^2 + y^2 + 2x + 2y = 27, \\ xy = -12. \end{cases}$

25. $\begin{cases} x^2 + y^2 - 5x - 5y = -4, \\ xy = 5. \end{cases}$

26. $\begin{cases} (7 + x)(6 + y) = 80, \\ x + y = 5. \end{cases}$

27. $\begin{cases} (x - 4)^2 + (y + 4)^2 = 100, \\ x + y = 14. \end{cases}$

28. $\begin{cases} xy + y + x = 17, \\ x^2y^2 + y^2 + x^2 = 129. \end{cases}$

29. $\begin{cases} b + a^2 = 5(a - b), \\ a + b^2 = 2(a - b). \end{cases}$

30. $\begin{cases} (13x)^2 + 2y^2 = 177, \\ (2y)^2 - 13x^2 = 3. \end{cases}$

31. $\begin{cases} \left(\dfrac{9}{x}\right)^2 = \left(\dfrac{25}{y}\right)^2 - 16, \\ \dfrac{9}{x^2} = \dfrac{25}{y^2}. \end{cases}$

32. $\begin{cases} x^2 + y^2 = 20, \\ 5x^2 - 3y^2 = 28. \end{cases}$

33. $\begin{cases} x^2 = -5 - 3xy, \\ 2xy = y^2 - 24. \end{cases}$

34. $\begin{cases} x + y + \sqrt{x + y} = 12, \\ x^3 + y^3 = 189. \end{cases}$

35. $\begin{cases} x^4 + x^2y^2 + y^4 = 133, \\ x^2 - xy + y^2 = 7. \end{cases}$

36. $\begin{cases} x + xy + y = 29, \\ x^2 + xy + y^2 = 61. \end{cases}$

37. $\begin{cases} 2x^2 - 5xy + 3x - 2y = 22, \\ 5xy + 7x - 8y - 2x^2 = 8. \end{cases}$

38. $\begin{cases} x + y = 74, \\ x^2 + y^2 = 3026. \end{cases}$

39. $\begin{cases} 7y^2 - 5x^2 + 20x + 13y = 29, \\ 5(x - 2)^2 - 7y^2 - 17y = -17. \end{cases}$

40. $\begin{cases} (3x+4y)(7x-2y)+3x+4y=44, \\ (3x+4y)(7x-2y)-7x+2y=30. \end{cases}$

41. $\begin{cases} x+y=4, \\ x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5=364. \end{cases}$

42. $\begin{cases} x^5-y^5=31, \\ x-y=1. \end{cases}$

43. $\begin{cases} x^4+y^4=82, \\ x^2+y^2+2x^2y^2=28. \end{cases}$

44. $\begin{cases} x^2+y^2-xy=80, \\ x-y-xy=-8. \end{cases}$

45. $\begin{cases} 8a+8b-ab-a^2=18, \\ 5a+5b-b^2-ab=24. \end{cases}$

46. $\begin{cases} (x^3+x^2y+xy^2+y^3)(x+y)=325, \\ (x^3-x^2y+xy^2-y^3)(x-y)=13. \end{cases}$

47. $\begin{cases} 2(x+4)^2-5(y-7)^2=75, \\ 7(x+4)^2+15(y-7)^2=1075. \end{cases}$

48. $\begin{cases} x^3+y^3=(a+b)(x-y), \\ x^2-xy+y^2=a-b. \end{cases}$

HIGHER EQUATIONS INVOLVING QUADRATICS

151. An equation of a degree above the second may often be reduced to the solution of a quadratic after applying the factor theorem. See § 92.

Example. Solve $\quad 2x^3+x^2-10x+7=0. \qquad (1)$

By the factor theorem, $x-1$ is found to be a factor,

giving $\quad (x-1)(2x^2+3x-7)=0. \qquad (2)$

Hence by § 22, $x-1=0$ and $2x^2+3x-7=0. \qquad (3)$

From $x-1=0$, $\quad x=1. \qquad (4)$

From $2x^2+3x-7=0$, $\quad x=\dfrac{-3\pm\sqrt{65}}{4}. \qquad (5)$

Hence (4) and (5) give the three roots of (1).

EXERCISES

Solve each of the following equations:

1. $7x^3 - 11x^2 + 4x = 0.$
2. $3x^4 + x^3 + 2x^2 + 24x = 0.$
3. $3x^3 - 16x^2 + 23x - 6 = 0.$
4. $5x^3 + 2x^2 + 4x = -7.$
5. $28x^3 - 10x^2 - 44x = 6.$
6. $x^4 - 3x^3 + 3x^2 - x = 0.$
7. $4x^3 + 12x^2 - 3x - 9 = 0.$
8. $x^4 - 5x^3 + 2x^2 + 20x = 24.$
9. $6x^3 + 29x^2 - 19x = 16.$
10. $15x^4 + 49x^3 - 92x^2 + 28x = 0.$

EQUATIONS IN THE FORM OF QUADRATICS

152. If an equation of higher degree contains a certain expression and also the square of this expression, and involves the unknown in no other way, then the equation is a **quadratic in the given expression.**

Ex. 1. Solve $$x^4 + 7x^2 = 44. \qquad (1)$$

This may be written, $(x^2)^2 + 7(x^2) = 44, \qquad (2)$

which is a *quadratic in* x^2. Solving, we find

$$x^2 = 4 \text{ and } x^2 = -11. \qquad (3)$$

Hence, $$x = \pm 2 \text{ and } x = \pm\sqrt{-11}. \qquad (4)$$

Ex. 2. Solve $$x + 2 + 3\sqrt{x+2} = 18. \qquad (1)$$

Since $x + 2$ is the square of $\sqrt{x+2}$, this is a quadratic in $\sqrt{x+2}$.

Solving we find $$\sqrt{x+2} = 3 \text{ and } \sqrt{x+2} = -6. \qquad (2)$$

Hence $$x + 2 = 9 \text{ and } x + 2 = 36, \qquad (3)$$

Whence $$x = 7 \text{ and } x = 34. \qquad (4)$$

Ex. 3. Solve $(2x^2 - 1)^2 - 5(2x^2 - 1) - 14 = 0.$

First solve as a quadratic in $2x^2 - 1$ and then solve the two resulting quadratics in x.

Ex. 4. Solve $x^2 - 7x + 40 - 2\sqrt{x^2 - 7x + 69} = -26.$ (1)

Add 29 to each member, obtaining

$$x^2 - 7x + 69 - 2\sqrt{x^2 - 7x + 69} = 3. \qquad (2)$$

Solve (2) as a quadratic in $\sqrt{x^2 - 7x + 69}$, obtaining

$$\sqrt{x^2 - 7x + 69} = 3 \text{ and } \sqrt{x^2 - 7x + 69} = -1, \qquad (3)$$

whence $$x^2 - 7x + 69 = 9 \text{ or } 1. \qquad (4)$$

The solution of the two quadratics in (4) will give the four values of x satisfying (1).

EXERCISES

Solve the following equations:

1. $x^6 + 2x^3 = 80.$
2. $5x - 4 - 2\sqrt{5x - 4} = 63.$
3. $(2 - x + x^2)^2 + x^2 - x = 18.$
4. $a^2 - 3a + 4 - 3\sqrt{a^2 - 3a + 4} = -2.$
5. $3a^6 - 7a^3 - 1998 = 0.$
6. $x^2 - 8x + 16 + 6\sqrt{x^2 - 8x + 16} = 40.$
7. $\left(a + \frac{2}{a}\right)^2 + 4\left(a + \frac{2}{a}\right) = 21.$
8. $a^8 - 97a^4 + 1296 = 0.$
9. $a^2 - 3a + 4 + \sqrt{a^2 - 3a + 15} = 19.$
10. $(5x - 7 + 3x^2)^2 + 3x^2 + 5x - 247 = 0.$
11. $\sqrt[3]{7x - 6} - 4\sqrt[6]{7x - 6} + 4 = 0.$

RELATIONS BETWEEN THE ROOTS AND THE COEFFICIENTS OF A QUADRATIC

153. If in the general quadratic, $ax^2 + bx + c = 0$, we divide both members by a and put $\frac{b}{a} = p$, $\frac{c}{a} = q$, we have $x^2 + px + q = 0$.

Solving, $x_1 = \dfrac{-p + \sqrt{p^2 - 4q}}{2}$, and $x^2 = \dfrac{-p - \sqrt{p^2 - 4q}}{2}$.

Adding x_1 and x_2, $$x_1 + x_2 = -\frac{2p}{2} = -p. \qquad (1)$$

Multiplying x_1 and x_2, $$x_1x_2 = \frac{p^2 - (p^2 - 4q)}{4} = q. \qquad (2)$$

Hence in a quadratic of the form $x^2 + px + q = 0$, the sum of the roots is $-p$, and the product of the roots is q.

The expression $p^2 - 4q = \dfrac{b^2}{a^2} - \dfrac{4c}{a} = \dfrac{b^2 - 4ac}{a^2}$.

Hence $p^2 - 4q$ is positive, negative, or zero, according as $b^2 - 4ac$ is positive, negative, or zero.

Hence, as found on pp. 87, 89, the roots of

$$ax^2 + bx + c = 0, \text{ or } x^2 + px + q = 0 \text{ are:}$$

real and distinct, if $b^2 - 4ac > 0$, *or* $p^2 - 4q > 0$, (3)

real and equal, if $b^2 - 4ac = 0$, *or* $p^2 - 4q = 0$, (4)

imaginary, if $b^2 - 4ac < 0$, *or* $p^2 - 4q < 0$. (5)

By means of (1) to (5), we may determine the character of the roots of a quadratic without solving it.

Ex. 1. Determine the character of the roots of

$$8x^2 - 3x - 9 = 0.$$

Since $b^2 - 4ac = 9 - 4 \cdot 8(-9) = 297 > 0$, the roots are real and distinct. Since $b^2 - 4ac$ is not a perfect square, the roots are irrational.

Since $q = -\frac{9}{8} = x_1x_2$, the roots have *opposite* signs.

Since $p = -\frac{3}{8}$ or $-p = \frac{3}{8} = x_1 + x_2$, the positive root is *greater* in absolute value.

Ex. 2. Examine $3x^2 + 5x + 2 = 0$.

Since $b^2 - 4ac = 25 - 4 \cdot 3 \cdot 2 = 1 > 0$, the roots are *real* and *distinct*.

Since $b^2 - 4ac$ is a perfect square, the roots are rational.

Since $q = \frac{2}{3} = x_1x_2$, the roots have the *same* sign.

Since $-p = -\frac{5}{3} = x_1 + x_2$, the roots are both *negative*.

Ex. 3. Examine $x^2 - 14x + 49 = 0$.

Since $p^2 - 4q = 196 - 4 \cdot 49 = 0$, the roots are real and coincident.

Ex. 4. Examine $x^2 - 7x + 15 = 0$.

Since $p^2 - 4q = 49 - 4 \cdot 15 = -11$, the roots are imaginary.

EXERCISES

Without solving, determine the character of the roots in each of the following:

1. $5x^2 - 4x - 5 = 0.$
2. $6x^2 + 4x + 2 = 0.$
3. $x^2 - 4x + 8 = 0.$
4. $2 + 2x^2 = 4x.$
5. $6x + 8x^2 = 9.$
6. $1 - a^2 = 3a.$
7. $6a - 30 = 3a^2.$
8. $6a^2 + 6 = 13a.$
9. $16m^2 + 4 = 16m.$
10. $25a^2 - 10a = 8.$
11. $20 - 13b - 15b^2 = 0.$
12. $10y^2 + 39y + 14 = 0.$
13. $3a^2 + 5a + 22.$
14. $3a^2 - 22a + 21 = 0.$
15. $5b^2 + 6b = 27.$
16. $6a - 17 = 11a^2.$

FORMATION OF EQUATIONS WHOSE ROOTS ARE GIVEN

154. Ex. 1. Form the equation whose roots are 7 and -4.

From (1) and (2), § 153, we have

$$x_1 + x_2 = -p = 7 + (-4) = 3. \quad \text{Hence } p = -3.$$

And $\qquad x_1x_2 = q = 7(-4) = -28.$

Hence $\qquad x^2 + px + q = 0$ becomes $x^2 - 3x - 28 = 0.$

In case the equation is to have more than two roots, we proceed as in the following example:

Ex. 2. Form the equation whose roots are 2, 3, and 5.

Recalling the solution by factoring, we may write the desired equation in the factored form as follows:

$$(x-2)(x-3)(x-5) = 0.$$

Obviously 2, 3, and 5, are the roots and the only roots of this equation. Hence the desired equation is:

$$(x-2)(x-3)(x-5) = x^3 - 10x^2 + 31x - 30 = 0.$$

EXERCISES

Form the equations whose roots are:

1. $3, -7$.
2. b, c.
3. $a, -b, -c$.
4. $5, -4, -2$.
5. $\sqrt{5}, -\sqrt{5}$.
6. $a-\sqrt{3}, a+\sqrt{3}$.
7. $-5, -6$.
8. $-b+k, -b-k$.
9. $\sqrt{-1}, -\sqrt{-1}$.
10. $a, -b$.
11. $8+\sqrt{3}, 8-\sqrt{3}$.
12. $2, 3, 4, 5$.
13. $3+2\sqrt{-1}, 3-2\sqrt{-1}$.
14. $5-\sqrt{-1}, 5+\sqrt{-1}$.
15. $1, \frac{1}{2}, \frac{1}{3}, 3$.
16. $\dfrac{-b+\sqrt{b^2-4ac}}{2a}, \dfrac{-b-\sqrt{b^2-4ac}}{2a}$.

155. An expression of the second degree in a single letter may be **resolved into factors**, each of the first degree in that letter, by solving a quadratic equation.

Ex. 1. Factor $6x^2-17x+5$.

This trinomial may be written, $6(x^2-\frac{17}{6}x+\frac{5}{6})$.

Solving the equation, $x^2-\frac{17}{6}x+\frac{5}{6}=0$, we find $x_1=\frac{1}{3}$ and $x_2=\frac{5}{2}$. Hence by the factor theorem, § 92, $x-\frac{1}{3}$ and $x-\frac{5}{2}$ are factors of $x^2-\frac{17}{6}x+\frac{5}{6}$. And finally

$$6(x^2-17x+5)=6(x-\tfrac{1}{3})(x-\tfrac{5}{2})=3(x-\tfrac{1}{3})\cdot 2(x-\tfrac{5}{2})$$
$$=(3x-1)(2x-5).$$

This process is not needed when the factors are *rational*, but it is applicable equally well when the factors are *irrational* or *imaginary*.

Ex. 2. Factor $3x^2+8x-7=3(x^2+\frac{8}{3}x-\frac{7}{3})$.

Solving the equation $x^2+\frac{8}{3}x-\frac{7}{3}=0$, we find,

$$x_1=\frac{-4+\sqrt{37}}{3} \text{ and } x_2=\frac{-4-\sqrt{37}}{3}.$$

Hence as above:

$$3x^2+8x-7=3\left[x-\frac{-4+\sqrt{37}}{3}\right]\left[x-\frac{-4-\sqrt{37}}{3}\right]$$
$$=3\left[x+\frac{4}{3}-\frac{\sqrt{37}}{3}\right]\left[x+\frac{4}{3}+\frac{\sqrt{37}}{3}\right].$$

EXERCISES

In exercises 1 to 16, p. 110, transpose all terms of each equation to the first member, and then factor this member.

PROBLEMS INVOLVING QUADRATIC EQUATIONS

In each of the following problems, interpret both solutions of the quadratic involved:

1. The area of a rectangle is 2400 square feet and its perimeter is 200 feet. Find the length of its sides.

2. The area of a rectangle is a square feet and its perimeter is $2b$ feet. Find the length of its sides. Solve 1 by substitution in the formula thus obtained.

3. A picture measured inside the frame is 18 by 24 inches. The area of the frame is 288 square inches. Find its width.

4. If in problem 3 the sides of the picture are a and b and the area of the frame c, find the width of the frame.

5. The sides a and b of a right triangle are increased by the same amount, thereby increasing the square on the hypotenuse by $2k$. Find by how much each side is increased.

Make a problem which is a special case of this and solve it by substitution in the formula just obtained.

6. The hypotenuse c and one side a are each increased by the same amount, thereby increasing the square on the other side by $2k$. Find how much was added to the hypotenuse.

Make a problem which is a special case of this and solve it by substituting in the formula just obtained.

7. A rectangular park is 80 by 120 rods. Two driveways of equal width, one parallel to the longer and one to the shorter side, run through the park. What is the width of the driveways if their combined area is 591 square rods?

8. If in problem 7 the park is a rods wide and b rods long and the area of the driveways is c square rods, find their width.

9. The diagonal of a rectangle is a and its perimeter $2b$. Find its sides.

Make a problem which is a special case of this and solve it by substituting in the formula just obtained.

10. If in problem 9 the difference between the length and width is b and the diagonal is a, find the sides. Show how one solution can be made to give the results for both problems 9 and 10.

11. Find two consecutive integers whose product is a.

Make a problem which is a special case of this and solve it by substituting in the formula just obtained.

What special property must a have in order that this problem may be possible. Answer this from the formula.

12. A rectangular sheet of tin, 12 by 16 inches, is made into an open box by cutting out a square from each corner and turning up the sides. Find the size of the square cut out if the volume of the box is 180 cubic inches.

The resulting equation is of the third degree. Solve it by factoring. See § 151. Obtain three results and determine which are applicable to the problem.

13. A square piece of tin is made into an open box containing a cubic inches, by cutting from each corner a square whose side is b inches and then turning up the sides. Find the dimensions of the original piece of tin.

14. A rectangular piece of tin is a inches longer than it is wide. By cutting from each corner a square whose side is b inches and turning up the sides, an open box containing c cubic inches is formed. Find the dimensions of the original piece of tin.

15. The hypotenuse of a right triangle is 20 inches longer than one side and 10 inches longer than the other. Find the dimensions of the triangle.

16. If in problem 15 the hypotenuse is a inches longer than one side and b inches longer than the other, find the dimensions of the triangle.

17. The area of a circle exceeds that of a square by 10 square inches, while the perimeter of the circle is 4 less than that of the square. Find the side of the square and the radius of the circle.

Use $3\frac{1}{7}$ as the value of π.

18. If in problem 17 the area of the circle exceeds that of the square by a square inches, while its perimeter is $2b$ inches less than that of the square, find the dimensions of the square and the circle.

Determine from this general solution under what conditions the problem is possible.

19. Find three consecutive integers such that the sum of their squares is a.

Make a problem which is a special case of this and solve it by means of the formula just obtained. From the formula discuss the cases, $a = 2$, $a = 5$, $a = 14$. Find another value of a for which the problem is possible.

20. The difference of the cubes of two consecutive integers is 397. Find the integers.

21. The upper base of a trapezoid is 8 and the lower base is 3 times the altitude. Find the altitude and the lower base if the area is 78.

See problem 7, p. 48.

22. The lower base of a trapezoid is 4 greater than twice the altitude, and the upper base is $\frac{1}{2}$ the lower base. Find the two bases and the altitude if the area is $52\frac{1}{2}$.

23. The lower base of a trapezoid is twice the upper, and its area is 72. If $\frac{1}{2}$ the altitude is added to the upper base, and the lower is increased by $\frac{1}{4}$ of itself, the area is then 120. Find the dimensions of the trapezoid.

24. The upper base of a trapezoid is equal to the altitude, and the area is 48. If the altitude is decreased by 4, and the upper base by 2, the area is then 14. Find the dimensions of the trapezoid.

25. The upper base of a trapezoid is 4 more than $\frac{1}{2}$ the lower base, and the area is 84. If the upper base is decreased by 5, and the lower is increased by $\frac{1}{2}$ the altitude, the area is 78. Find the dimensions of the trapezoid.

26. The area of an equilateral triangle multiplied by $\sqrt{3}$, plus 3 times its perimeter, equals 81. Find the side of the triangle.

See problem 15, p. 236, E. C.

27. The area of a regular hexagon multiplied by $\sqrt{3}$, minus twice its perimeter, is 504. Find the length of its side.

See problem 20, p. 237, E. C.

28. If a times the perimeter of a regular hexagon, plus $\sqrt{3}$ times its area, equals b, find its side.

29. The perimeter of a circle divided by π, plus $\sqrt{3}$ times the area of the inscribed regular hexagon, equals $78\frac{3}{4}$. Find the radius of the circle.

30. The area of a regular hexagon inscribed in a circle plus the perimeter of the circle is a. Find the radius of the circle.

31. One edge of a rectangular box is increased 6 inches, another 3 inches, and the third is decreased 4 inches, making a cube whose volume is 862 cubic inches greater than that of the original box. Find its dimensions.

32. Of two trains one runs 12 miles per hour faster than the other, and covers 144 miles in one hour less time. Find the speed of each train.

In a township the main roads run along the section lines, one half of the road on each side of the line.

33. Find the area included by the main roads of a township if they are 4 rods wide.

34. If the area included by the main roads of a township is 11,196 square rods, find the width of the roads.

35. Find the width of the roads in problem 34 if the area included by them is a square rods.

CHAPTER VIII

ALGEBRAIC FRACTIONS

156. An **algebraic fraction** is the indicated quotient of two algebraic expressions.

Thus $\frac{n}{d}$ means *n divided by d.*

From the definition of a fraction and § 11, it follows that the *product of a fraction and its denominator equals its numerator.*

That is,
$$d \cdot \frac{n}{d} = n.$$

REDUCTION OF FRACTIONS

157. The **form** of a fraction may be modified in various ways without changing its **value.** Any such transformation is called a **reduction of the fraction.**

The most important reductions are the following:

(*A*) *By manipulation of signs.*

E.g. $\frac{n}{d} = -\frac{-n}{d} = -\frac{n}{-d} = \frac{-n}{-d}$; $\frac{b-a}{c-d} = -\frac{a-b}{c-d} = \frac{a-b}{d-c}$.

(*B*) *To lowest terms.*

E.g. $\frac{x^4+x^2+1}{x^6-1} = \frac{(x^2+x+1)(x^2-x+1)}{(x-1)(x^2+x+1)(x+1)(x^2-x+1)}$
$$= \frac{1}{(x-1)(x+1)}.$$

(*C*) *To integral or mixed expressions.*

E.g. $\frac{2x^3+x^2+x+2}{x^2+1} = 2x+1+\frac{-x+1}{x^2+1} = 2x+1-\frac{x-1}{x^2+1}$.

(*D*) *To equivalent fractions having a common denominator.*

E.g. $\frac{2}{x+3}$ and $\frac{3}{x+2}$ become respectively $\frac{2(x+2)}{(x+3)(x+2)}$ and $\frac{3(x+3)}{(x+3)(x+2)}$; $a+1$ and $\frac{1}{a-1}$ become respectively $\frac{a^2-1}{a-1}$ and $\frac{1}{a-1}$.

158. These reductions are useful in connection with the various operations upon fractions. They depend upon the principles indicated below.

Reduction (*A*) is simply an application of the law of signs in division, § 28. It is often needed in connection with reduction (*D*). See § 159.

Reduction (*B*) depends upon the theorem, § 47, $\frac{ak}{bk}=\frac{a}{b}$, by which *a common factor may be removed from both terms of a fraction.* It is useful in keeping expressions *simplified.* This reduction is *complete* when numerator and denominator have been divided by their H. C. F. See §§ 95–102.

Reduction (*C*) is merely the process of performing the indicated division, the result being *integral* when the division is *exact*, otherwise a *mixed expression.*

In case there is a *remainder* after the division has been carried as far as possible, this part of the quotient can only be *indicated.*

Thus
$$\frac{D}{d}=q+\frac{R}{d},$$

in which D is dividend, d is divisor, q is quotient, and R is remainder.

Reduction (*D*) depends upon the theorem of § 47, $\frac{a}{b}=\frac{ka}{kb}$, by which *a common factor is introduced into the terms of a fraction.*

A fraction is thus reduced to another fraction whose denominator is *any required multiple* of the given denominator.

If two or more fractions are to be reduced to equivalent fractions having a common denominator, this denominator must be a *common multiple* of the given denominators, and for simplicity the L. C. M. is used.

EXERCISES

Reduce the following so that the letters in each factor shall occur in alphabetical order, and no negative sign shall stand before a numerator or denominator, or before the first term of any factor.

1. $\dfrac{n-m}{b-a}$.

2. $-\dfrac{(b-a)(c-d)}{x(s-r-t)}$.

3. $\dfrac{-(x-y)}{(b-a)(c-d)}$.

4. $\dfrac{-(x-y)(z-y)}{-(b-a)(c-d)}$.

5. $\dfrac{r-s}{(a-b)(c-b)(c-a)}$.

6. $\dfrac{-a(c+b)}{b(c-a)}$.

7. $\dfrac{-(c-a)(d-c)}{(a-b)(b-c)}$.

8. $\dfrac{(b-a)(c-b)(c-a)}{(y-x)(y-z)(z-x)}$.

9. $-\dfrac{-1}{(a-b)(b-c)(c-a)}$.

10. $\dfrac{(c-b-a)(b-a-c)}{3(a-c)(b-c)(c-a)}$.

11. $\dfrac{(3c-2a)(4b-a)d}{(-a+b)(a-b)(c-a)}$.

12. $\dfrac{-(-r-s)(s-t)(t-r)}{(n-m)(-k-m-l)}$.

Reduce each of the following to lowest terms:

13. $\dfrac{a^4-b^4}{a^6-b^6}$.

14. $\dfrac{c^2-(a-b)^2}{(a+c)^2-b^2}$.

15. $\dfrac{7ax^2-56a^4x^5}{28x^2(1-64a^6x^6)}$.

16. $\dfrac{m^3+5m^2+7m+3}{m^2+4m+3}$.

17. $\dfrac{a^3-7a+6}{a^3-7a^2+14a-8}$.

18. $\dfrac{x^3+2x^2+2x+1}{x^4+x^3-x^2-2x-2}$.

19. $\dfrac{2x^3-x^2-8x-3}{2x^3-3x^2-7x+3}$.

20. $\dfrac{4x^3+8x^2-3x+5}{6x^3-5x^2+4x-1}$.

21. $\dfrac{x^2-xy+y^2+x-y+3}{x^3+y^3+x^2-y^2+3x+3y}$.

22. $\dfrac{a^4+a^2b^2+b^4+a^3+b^3}{a^2+ab+b^2+a+b}$.

23. $\dfrac{x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 - a^4}{x^2 + 2xy + y^2 - a^2}$.

24. $\dfrac{x^2y - x^2z + y^2z - xy^2 + xz^2 - yz^2}{x^2 - (y+z)x + yz}$.

25. $\dfrac{2x^4 - x^3 - 20x^2 + 16x - 3}{3x^4 + 5x^3 - 30x^2 - 41x + 5}$.

26. $\dfrac{3a^3 - 8a^2b - 5ab^2 + 6b^3}{a^3 + a^2b - 9ab^2 - 9b^3}$.

27. $\dfrac{2r^3 + r^2s + rs^2 + 2s^3}{2r^4 + r^3s + 3r^2s^2 + rs^3 + 2s^4}$.

Reduce each of the following to an integral or mixed expression:

28. $\dfrac{x^4+1}{x+1}$.

29. $\dfrac{x^5+1}{x+1}$.

30. $\dfrac{x^4}{x-1}$.

31. $\dfrac{a^3}{a^2+a+1}$.

32. $\dfrac{c^5}{c^3+c^2-c+1}$.

33. $\dfrac{x^2-x+1}{x^2+x+1}$.

34. $\dfrac{a^4+a^2b^2+b^4}{a-b}$.

35. $\dfrac{3a^3-3a^2+3a-1}{a-2}$.

36. $\dfrac{x^3-x^2-x+1}{x^3+x^2+x-1}$.

37. $\dfrac{4m^4-3m^3+3}{2m^2-2m+1}$.

Reduce each of the following sets of expressions to equivalent fractions having the lowest common denominator:

38. $\dfrac{1}{x^4-3x^2y^2+y^4},\ \dfrac{1}{x^2-xy-y^2},\ \dfrac{1}{x^2+xy-y^2}$.

39. $\dfrac{a+b}{5a^2c+12cd-6ad-10ac^2},\ \dfrac{a}{5ac-6d},\ \dfrac{b}{a-2c}$.

40. $\dfrac{x^2+y^2}{x^3+y^3+x^2-xy+y^2},\ \dfrac{x+y-1}{x^2-xy+y^2},\ \dfrac{x^2+xy+y^2}{x+y+1}$.

41. $\dfrac{x}{(a-b)(c-b)(c-a)}$, $\dfrac{y}{(a-b)(b-c)(a-c)}$, $\left[\dfrac{z}{(b-a)(b-c)(a-c)}\right.$.

42. $\dfrac{a}{b+c}$, $\dfrac{b}{c+a}$, $\dfrac{c}{a+b}$, d.

43. $\dfrac{b-c}{(a-c)(a-b)}$, $\dfrac{a-b}{(c-a)(b-c)}$, $\dfrac{c-a}{(b-a)(c-b)}$.

44. $\dfrac{m-n}{a^3-6a^2+11a-6}$, $\dfrac{a+2}{a^2-4a+3}$, $\dfrac{a+3}{a^2-3a+2}$.

If a, b, m are positive numbers, arrange each of the following sets in decreasing order. Verify the results by substituting convenient Arabic numbers for a, b, m.

Suggestion. Reduce the fractions in each set to equivalent fractions having a common denominator.

45. $\dfrac{a}{a+1}$, $\dfrac{2a}{a+2}$, $\dfrac{3a}{a+3}$.

46. $\dfrac{m}{2m+1}$, $\dfrac{2m}{3m+2}$, $\dfrac{3m}{4m+3}$.

47. $\dfrac{a+3b}{a+4b}$, $\dfrac{a+b}{a+2b}$, $\dfrac{a+4b}{a+5b}$.

48. Show that, for a different from zero, neither $\dfrac{n+a}{d+a}$ nor $\dfrac{n-a}{d-a}$ can equal $\dfrac{n}{d}$, unless $n=d$. State this result in words, and fix it in mind as an impossible reduction of a fraction.

ADDITION AND SUBTRACTION OF FRACTIONS

159. Fractions which have a common denominator are added or subtracted in accordance with the distributive law for division, § 30.

That is,
$$\frac{a}{d}+\frac{b}{d}-\frac{c}{d}=\frac{a+b-c}{d}.$$

In order to add or subtract fractions not having a common denominator, they should first be reduced to equivalent fractions having a common denominator.

When several fractions are to be combined, it is sometimes best to take only part of them at a time. In any case it is advantageous to keep all expressions in the factored form as long as possible.

Ex. $$\frac{1}{(x-1)(x-2)}-\frac{1}{(2-x)(x-3)}+\frac{1}{(3-x)(4-x)}.$$

Taking the first two together, we have

$$\frac{1}{(x-1)(x-2)}+\frac{1}{(x-2)(x-3)}=\frac{2x-4}{(x-1)(x-2)(x-3)}=\frac{2}{(x-1)(x-3)}.$$

Taking this result with the third,

$$\frac{2}{(x-1)(x-3)}+\frac{1}{(x-3)(x-4)}=\frac{3x-9}{(x-1)(x-3)(x-4)}=\frac{3}{(x-1)(x-4)}.$$

If all are taken at once, the work should be carried out as follows: The numerator of the sum is

$$(x-3)(x-4)+(x-1)(x-4)+(x-1)(x-2).$$

Adding the first two terms with respect to $(x-4)$, we have

$$2(x-2)(x-4)+(x-1)(x-2).$$

Adding these with respect to $(x-2)$, we have $3(x-3)(x-2)$.

Hence the sum is $\dfrac{3(x-3)(x-2)}{(x-1)(x-2)(x-3)(x-4)}=\dfrac{3}{(x-1)(x-4)}.$

EXERCISES

Perform the following indicated additions and subtractions:

1. $\dfrac{2}{x-3}+\dfrac{3}{x-4}-\dfrac{4}{x-5}.$

2. $\dfrac{3}{4(x+3)}-\dfrac{5}{8(x+5)}-\dfrac{1}{8(x+1)}.$

3. $\dfrac{1}{2(x-1)}-\dfrac{4}{x-2}+\dfrac{7}{2(x-3)}.$

4. $\dfrac{1}{12(x+1)}-\dfrac{7}{3(x-2)}+\dfrac{13}{4(x-3)}.$

5. $\dfrac{2}{(x+1)^2}+\dfrac{3}{x+1}+\dfrac{4}{x-2}$.

6. $\dfrac{1}{3(x+1)}-\dfrac{x-2}{3(x^2-x+1)}$.

7. $\dfrac{5x+6}{x^2+x+1}-\dfrac{3x-4}{x^2-x+1}$.

8. $\dfrac{1}{5(x+2)}+\dfrac{4x-8}{5(x^2+1)}$.

9. $\dfrac{2}{(x-2)^2}-\dfrac{1}{x-2}+\dfrac{1}{x+1}$.

10. $\dfrac{2}{(x-2)^2}+\dfrac{1}{5(x-2)}-\dfrac{x+2}{5(x^2+1)}$.

11. $\dfrac{1}{(x-1)^2}+\dfrac{1}{(x-1)}-\dfrac{1}{(x^2-1)}$.

12. $\dfrac{1}{2(1-3x)^3}+\dfrac{3}{8(1-3x)^2}+\dfrac{3}{32(1-3x)}+\dfrac{1}{32(1+x)}$.

13. $\dfrac{1}{(1-a)(2-a)}-\dfrac{1}{(2-a)(a-3)}+\dfrac{2}{(3-a)(a-1)}$.

14. $\dfrac{xy}{(z-y)(x-z)}-\dfrac{yz}{(x-z)(x-y)}-\dfrac{xz}{(y-x)(y-z)}$.

15. $\dfrac{1}{a-1}-\dfrac{2a-5}{a^2-2a+1}-\dfrac{5a^2-3a-2}{(a-1)^3}$.

16. $\dfrac{1}{m^2+m+1}-\dfrac{1}{m^2-m+1}+\dfrac{2m+2}{m^4+m^2+1}$.

17. $\dfrac{1}{b^2-3b+2}+\dfrac{1}{b^2-5b+6}-\dfrac{2}{b^2-4b+3}$.

18. $\dfrac{r+s}{(r-t)(s-t)}-\dfrac{s+t}{(r-s)(t-r)}-\dfrac{r+t}{(t-s)(s-r)}$.

19. $\dfrac{p^2+q^2}{(p-q)(p+r)}+\dfrac{q^2-pr}{(q-r)(q-p)}+\dfrac{r^2+pq}{(r-q)(r+p)}$.

20. $\dfrac{3x^2+1}{5x^2-18x+9}-\dfrac{2x^2+2}{4x^2-11x-3}$.

MULTIPLICATION AND DIVISION OF FRACTIONS

160. Theorem. *The product of two fractions is a fraction whose numerator is the product of the given numerators and whose denominator is the product of the given denominators.*

Proof. We are to prove that $\frac{a}{b} \cdot \frac{n}{d} = \frac{an}{bd}$.

Let $$x = \frac{a}{b} \cdot \frac{n}{d}.$$

Then $$bdx = bd\left(\frac{a}{b} \cdot \frac{n}{d}\right). \qquad \S\ 7$$

$$bdx = b \cdot \frac{a}{b} \cdot d \cdot \frac{n}{d}. \qquad \S\ 8$$

$$bdx = an. \qquad \S\ 11$$

Hence, $$x = \frac{an}{bd}.$$

Therefore, $$\frac{a}{b} \cdot \frac{n}{d} = \frac{an}{bd}. \qquad \S\ 2$$

Corollary 1. *A fraction is raised to any power by raising numerator and denominator separately to that power.*

For by the above theorem, $\frac{a}{b} \cdot \frac{a}{b} = \frac{a^2}{b^2}$, $\frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} = \frac{a^3}{b^3}$, etc.

Corollary 2. *A fraction multiplied by itself inverted equals* $+1$.

For $\frac{n}{d} \cdot \frac{d}{n} = \frac{nd}{nd} = +1$ and $-\frac{n}{d} \cdot \left(-\frac{d}{n}\right) = \frac{nd}{nd} = +1$.

161. Definitions. If the product of two numbers is $+1$, each is called the **reciprocal** of the other. Hence from Cor. 2, the reciprocal of a fraction is the fraction inverted.

Also, since from $ab = 1$ we have $a = \frac{1}{b}$ and $b = \frac{1}{a}$, it follows that if two numbers are reciprocals of each other, then either one is the quotient obtained by dividing 1 by the other.

162. Theorem. *To divide by any number is equivalent to multiplying by its reciprocal.*

Proof. We are to prove that $n \div d$ or $\frac{n}{d} = n \cdot \frac{1}{d}$. This is an immediate consequence of § 29.

Corollary 1. *To divide a number by a fraction is equivalent to multiplying by the fraction inverted.*

For by § 161 the reciprocal of the fraction is the fraction inverted.

Corollary 2. *A fraction is divided by an integer by multiplying its denominator or dividing its numerator by that integer.*

For $$\frac{n}{d} \div a = \frac{n}{d} \cdot \frac{1}{a} = \frac{n}{ad}. \qquad \text{Cor. 1 and § 160}$$

and $$\frac{n}{d} \div a = \frac{n \div a}{d}, \text{ since } \frac{n}{ad} = \frac{n \div a}{d} \text{ by § 17.}$$

In multiplying and dividing fractions their terms should at once be put into *factored* forms.

When mixed expressions or sums of fractions are to be multiplied or divided, these operations are indicated by means of parentheses, and the additions or subtractions within the parentheses should be performed first, § 38.

Ex. Simplify $\left[\left(1 - a + \frac{2a^2}{1+a}\right) \div \left(\frac{1}{1+a} - \frac{1}{1-a}\right)\right] \cdot \frac{3a^3}{a^4-1}.$

Performing the indicated operations within the parentheses, we have

$$\left[\frac{1+a^2}{1+a} \div \frac{2a}{a^2-1}\right] \cdot \frac{3a^3}{a^4-1} = \frac{1+a^2}{1+a} \cdot \frac{a^2-1}{2a} \cdot \frac{3a^3}{(a^2-1)(a^2+1)} = \frac{3a^2}{2(a+1)}.$$

EXERCISES

Perform the following indicated operations and reduce each result to its simplest form.

1. $\frac{x^4 + x^2y^2 + y^4}{x^3 - y^3} \cdot \frac{x^2 - y^2}{x^3 + y^3}.$

2. $\dfrac{a^2 - b^2x^2 + acx^2 - bcx^3}{36\,a^4 - 9\,a^2 + 24\,a - 16} \div \dfrac{-ay^2 - bxy^2 - cx^2y^2}{20\,x^2 - 15\,ax^2 - 30\,a^2x^2}.$

3. $\dfrac{20\,r^2s^2 + 23\,rst - 21\,t^2}{8\,m^2n^3 - 48\,m^2n^2y + 72\,m^2ny^2} \times \dfrac{12\,mn^3 - 28\,mn^2y - 24\,mny^2}{10\,r^2s^2 + 24\,rst - 18\,t^2}$

4. $\left(\dfrac{a}{b} - \dfrac{b}{a}\right)\left(\dfrac{a^2}{b^2} + \dfrac{b^2}{a^2} - \dfrac{a}{b} - \dfrac{b}{a} + 1\right) \div \dfrac{a^5 + b^5}{a - b}.$ $\left[\div \dfrac{3\,n + 2\,y}{2\,s + 3\,t}.\right.$

5. $\left(\dfrac{1}{x} + \dfrac{1}{y}\right)\left(\dfrac{1}{x} - \dfrac{1}{y}\right)\left(1 - \dfrac{x - y}{x + y}\right)\left(2 + \dfrac{2\,y}{x - y}\right).$

6. $\left(\dfrac{a}{a - b} - \dfrac{b}{a + b}\right)\left(\dfrac{1}{a^2} - \dfrac{1}{b^2}\right) \div \left(\dfrac{1}{a^2} + \dfrac{1}{b^2}\right).$

7. $\left(1 + \dfrac{b}{a - b}\right)\left(1 - \dfrac{b}{a + b}\right) \div \left(1 + \dfrac{b^2}{a^2 - b^2}\right).$

8. $\left(\dfrac{m + n}{m - n} - \dfrac{m - n}{m + n}\right)\left(m + n + \dfrac{2\,n^2}{m - n}\right) \div \left(\dfrac{m + n}{m - n} + \dfrac{m - n}{m + n}\right).$

9. $\left(\dfrac{x^2 + y^2}{x^2 - y^2} - \dfrac{x^2 - y^2}{x^2 + y^2}\right) \cdot \left(x^2 + y^2 + \dfrac{2\,x^2y^2 + 2\,y^4}{x^2 - y^2}\right) \div \left(\dfrac{x + y}{x - y} + \dfrac{x - y}{x + y}\right).$

10. $\left(\dfrac{x + y + z}{x + y} + \dfrac{z^2}{(x + y)^2}\right) \cdot \left(\dfrac{(x + y)^3}{(x + y)^3 - z^3}\right) \cdot \left(1 + \dfrac{z}{x + y}\right).$

11. $\dfrac{a^2 + ab + b^2}{a^2 - ab + b^2} \cdot \dfrac{a + b}{a^3 - b^3} \cdot \left(a^2 + \dfrac{b^3 - a^2b}{a + b}\right).$

12. $\dfrac{m^2 + mn}{m^2 + n^2} \cdot \dfrac{m^3 - mn^2 - m^2 + n^2}{m^3n - n^4} \cdot \dfrac{m^2n^2 + mn^3 + n^4}{m^4 - 2\,m^3 + m^2}$

$\div \dfrac{m^3n + 2\,m^2n^2 + mn}{m^4 - n^4}.$

13. $\left(xy^9 + x^9y - \dfrac{2\,x^{11}y^3}{x^2y^2}\right) \div \left[\dfrac{x^2 + y^2}{x^2} \cdot \left(\dfrac{1}{y^2} - \dfrac{1}{z^2}\right) - \dfrac{x^2 + y^2}{y^2} \cdot \left(\dfrac{1}{x^2} - \dfrac{1}{z^2}\right)\right].$

COMPLEX FRACTIONS

163. A fraction which contains a fraction either in its numerator or in its denominator or in both is called a **complex fraction.**

Since every fraction is an indicated operation in division, any complex fraction may be simplified by performing the indicated division.

It is usually better, however, to remove all the minor denominators at once by multiplying both terms of the complex fraction by the least common multiple of all the minor denominators according to § 47.

For example, $$\frac{\frac{x}{3}+\frac{x}{2}}{\frac{2x^2}{3}-\frac{3}{2}} = \frac{\left(\frac{x}{3}+\frac{x}{2}\right)\cdot 6}{\left(\frac{2x^2}{3}-\frac{3}{2}\right)\cdot 6} = \frac{2x+3x}{4x^2-9} = \frac{5x}{4x^2-9}.$$

A complex fraction may contain another complex fraction in one of its terms.

E.g. $\dfrac{1}{a+\dfrac{a+1}{a+\dfrac{1}{a-1}}}$ has the complex fraction $\dfrac{a+1}{a+\dfrac{1}{a-1}}$

in its denominator. This latter fraction is first reduced by multiplying its numerator and denominator by $a-1$, giving

$$\frac{a+1}{a+\dfrac{1}{a-1}} = \frac{a^2-1}{a^2-a+1}.$$

Substituting this result in the given fraction, we have

$$\frac{1}{a+\dfrac{a+1}{a+\dfrac{1}{a-1}}} = \frac{1}{a+\dfrac{a^2-1}{a^2-a+1}} = \frac{a^2-a+1}{a^3+a-1}.$$

EXERCISES

Simplify each of the following,

1. $$\frac{\dfrac{m^2+mn}{m^2-n^2}}{\dfrac{m}{m-n}-\dfrac{n}{m+n}}.$$

2. $$\frac{\dfrac{a^4-b^4}{a^2-2ab+b^2}}{\dfrac{a^2+ab}{a-b}}.$$

3. $$\frac{\dfrac{x^5-3x^4y+3x^3y^2-x^2y^3}{x^3y-y^4}}{\dfrac{x^5-2x^4y+x^3y^2}{x^2y^2+xy^3+y^4}}.$$

4. $$\frac{\dfrac{1}{a+x}+\dfrac{1}{a-x}+\dfrac{2a}{a^2-x^2}}{\dfrac{1}{a+x}-\dfrac{1}{a-x}-\dfrac{2a}{a^2-x^2}}.$$

5. $$\frac{\dfrac{1}{a+x}+\dfrac{1}{a-x}+\dfrac{2a}{a^2+x^2}}{\dfrac{1}{a-x}-\dfrac{1}{a+x}+\dfrac{2x}{a^2+x^2}}.$$

6. $$\frac{m^2-mn+n^2-\dfrac{m^3-n^3}{m+n}}{m^2+mn+n^2+\dfrac{m^3+n^3}{m-n}}.$$

7. $$\frac{\dfrac{a-\dfrac{1}{a^2}}{a-2+\dfrac{1}{a}}}{\dfrac{a^2+1+\dfrac{1}{a^2}}{a-2+\dfrac{2}{a}-\dfrac{1}{a^2}}}.$$

8. $$\frac{1}{1-\dfrac{1}{1-\dfrac{1}{x}}}.$$

9. $$\frac{1}{1-\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{x}}}}$$

10. $$\frac{3}{3+\dfrac{3}{3+\dfrac{3}{3+\dfrac{3}{x}}}}.$$

EQUATIONS INVOLVING ALGEBRAIC FRACTIONS

164. In solving a fractional equation, it is usually convenient to **clear it of fractions**, that is, to transform it into an equivalent equation containing no fractions.

In case no denominator contains any unknown this may be done by multiplying both members by the L. C. M. of all the denominators, § 62.

When, however, the unknown appears in any denominator, multiplying by the L. C. M. of all the denominators *may or may not* introduce new roots, as shown in the following examples.

It may easily be shown, that multiplying an *integral* equation by any expression containing the unknown *always* introduces new roots.

Ex. 1. Solve $$\frac{2}{x-2}+\frac{1}{x-3}=2. \qquad (1)$$

Clearing of fractions by multiplying by $(x-2)(x-3)$, and simplifying, we have $$2x^2-13x+20=(x-4)(2x-5)=0. \qquad (2)$$

The roots of (2) are 4 and $2\frac{1}{2}$, both of which satisfy (1). Hence no new root was introduced by clearing of fractions.

Ex. 2. Solve $$\frac{1}{x-1}=\frac{1}{(x-1)(x-2)}. \qquad (1)$$

Clearing of fractions, we have,
$$x-2=1. \qquad (2)$$

The only root of (2) is $x=3$, which is also the only root of (1). Hence no new root was introduced.

Ex. 3. Solve $$\frac{x-2}{x^2-4}=1. \qquad (1)$$

Clearing of fractions and simplifying, we have,
$$x^2-x-2=(x-2)(x+1)=0. \qquad (2)$$

The roots of (2) are 2 and -1. Now $x=-1$ is a root of (1), but $x=2$ is *not*, since we are not permitted to make a substitution which reduces a denominator to zero, § 50. Hence a new root has been introduced and (1) and (2) are not equivalent.

If the fraction $\frac{x-2}{x^2-4}$ is first reduced to lowest terms, we have the equation
$$\frac{1}{x+2}=1. \qquad (3)$$

Clearing of fractions, $$x+2=1. \qquad (4)$$

Now (3) and (1) are equivalent, -1 being the only root of each.

Ex. 4. Solve $$\frac{4x}{x^2-1}-\frac{x+1}{x-1}=1. \qquad (1)$$

Clearing of fractions and simplifying,

$$x^2-x=x(x-1)=0. \qquad (2)$$

The roots of (2) are 0 and 1. $x=0$ satisfies (1), but $x=1$ does not, since it is not a permissible substitution in either fraction of (1). Hence a new root has been introduced.

165. Examples 3 and 4 illustrate the *only cases in which new roots can be introduced* by multiplying by the L. C. M. of the denominators.

This can be shown by proving certain important theorems, the results of which are here used in the following directions for solving fractional equations:

(1) Reduce all fractions to their lowest terms.

(2) Multiply both members by the least common multiple of the denominators.

(3) Reject any root of the *resulting* equation which reduces any denominator of the *given* equation to zero. The remaining roots will then satisfy both equations, and hence are the solutions desired.

If when each fraction is in its lowest terms the given equation contains no two which have a factor common to their denominators, then *no new root* can enter the resulting equation and none need to be rejected. See Ex. 1 and Ex. 3 after being reduced.

If, however, any two or more denominators have some common factor $x-a$, then $x=a$ *may or may not be a new root* in the resulting equation, but in any case it is the only possible *kind* of new root which can enter, and must be tested. Compare Exs. 2 and 4.

Ex. 5. Examine $$\frac{3x+7}{x^2+2x+11}+\frac{5x}{x^2+3x+2}-\frac{x+1}{x-1}=8.$$

Since each fraction is in its lowest terms and no two denominators contain a common factor, then clearing of fractions will give an equation equivalent to the given one.

Ex. 6. Examine $\dfrac{2x+3}{x^2+5x+6} - \dfrac{x-7}{x^2+3x+2} = 4.$

Each fraction is in its lowest terms, but the two denominators have the factor $x+2$ in common. Hence $x = -2$ is the *only possible* new root which can enter the resulting integral equation, but on trial it is found not to be a root. Hence the two equations are equivalent.

EXERCISES

Determine whether each of the following when cleared of fractions produces an equivalent equation, and solve each.

1. $\dfrac{3x^3+3}{3x^2-7x+3} = x - 7.$

2. $\dfrac{x^2+4x+4}{x^2-4} = 2x+3.$

3. $\dfrac{3}{2x^2-x-1} + \dfrac{5}{x^2-1} + \dfrac{1}{x+1} = 0.$

4. $\dfrac{2x}{2x-1} + \dfrac{x}{x+1} - \dfrac{3x}{x-1} = -1.$

5. $\dfrac{1}{3(x-1)} - \dfrac{1}{x^2-1} = \dfrac{1}{4}.$

6. $\dfrac{2a-1}{a} + \dfrac{1}{2} = \dfrac{3a}{3a-1}.$

7. $\dfrac{2}{x-a} + \dfrac{3}{x-b} = \dfrac{6}{x-c}.$

8. $\dfrac{1}{a-x} - \dfrac{1}{a+x} = -\dfrac{3+x^2}{a^2-x^2}.$

9. $a^2b - \dfrac{a+x}{b} = ab^2 - \dfrac{b+x}{a}.$

10. $b = \dfrac{x-a}{1-ax}.$

11. $\dfrac{2}{x-10} + 10 = x + \dfrac{2}{10-x}.$

12. $\dfrac{6-x}{x-4} + \dfrac{x-4}{6-x} = \dfrac{c}{d}.$

13. $\dfrac{a}{2a-1} + \dfrac{24}{4a^2-1} = \dfrac{2(a-4)}{2a+1} - \dfrac{1}{9}.$

14. $\dfrac{a}{a-1} + \dfrac{a-1}{a} = \dfrac{a^2+a-1}{a^2-a}.$

15. $\dfrac{1}{a^2-4} - \dfrac{3}{2-a} = 1 + \dfrac{1}{3(a+2)}.$

16. $\dfrac{ax+b}{a+bx}+\dfrac{cx+d}{c+dx}=\dfrac{ax-b}{a-bx}+\dfrac{cx-d}{c-dx}.$

17. $\dfrac{(a-x)(x-b)}{(a-x)-(x-b)}=x.$ 18. $\dfrac{x+m-2n}{x+m+2n}=\dfrac{n+2m-2x}{n-2m+2x}.$

19. $\dfrac{(a-x)^2-(x-b)^2}{(a-x)(x-b)}=\dfrac{4ab}{a^2-b^2}.$

20. $\dfrac{1+3x}{5+7x}-\dfrac{9-11x}{5-7x}=14\cdot\dfrac{(2x-3)^2}{25-49x^2}.$

21. $\dfrac{x+2a}{2b-x}+\dfrac{x-2a}{2b+x}=\dfrac{4ab}{4b^2-x^2}.$

22. $\dfrac{1}{x-2}+\dfrac{7x}{24(x+2)}=\dfrac{5}{x^2-4}.$

23. $\dfrac{x+a}{x-a}+\dfrac{x-a}{x+a}=\dfrac{2(a^2+1)}{(1+a)(1-a)}.$

24. $\dfrac{x-m}{x+m}=\dfrac{n-x}{n+x}.$ 25. $\dfrac{1}{x-a}-\dfrac{2a}{x^2-a^2}=b.$

26. $\dfrac{4}{3x+1}+\dfrac{4(3x-1)}{2x+1}=\dfrac{2x+1}{3x+1}.$

27. $\dfrac{2x+3}{2(2x-1)}+\dfrac{7-3x}{3x-4}+\dfrac{x-7}{2(x+1)}=0.$

28. $\dfrac{1}{a-b}+\dfrac{a-b}{x}=\dfrac{1}{a+b}+\dfrac{a+b}{x}.$

29. $\dfrac{1}{\dfrac{3(m+n)^2}{p^2x}-\dfrac{m+n}{p}}=\dfrac{p}{2(m+n)}.$

30. $\dfrac{y^2+2y-2}{y^2+5y+6}+\dfrac{y}{y+3}=\dfrac{y}{y+2}.$

31. $\dfrac{5}{2x+3}+\dfrac{7}{3x-4}=\dfrac{8x^2-13x-64}{6x^2+x-12}$.

32. $\dfrac{3a^2}{x^3-a^3}-\dfrac{1}{x-a}+\dfrac{a}{x^2+ax+a^2}=c$.

33. $\dfrac{1-2x}{3-4x}-\dfrac{5-6x}{7-8x}=\dfrac{8}{3}\cdot\dfrac{1-3x^2}{21-52x+32x^2}$.

34. $\dfrac{m-q}{x-n}+\dfrac{n-p}{x-q}=\dfrac{m-q}{x-p}+\dfrac{n-p}{x-m}$.

35. $\dfrac{3}{x-3}-\dfrac{2}{x-2}+\dfrac{8}{4x^2-20x+24}=0$.

36. $\dfrac{27}{x^3+27}-\dfrac{3}{x^2-3x+9}+\dfrac{1}{x+3}=0$.

37. $\dfrac{x-9}{x-5}-\dfrac{x-7}{x-2}-\dfrac{x-9}{x-4}=\dfrac{x-8}{x-5}-\dfrac{x-7}{x-4}-\dfrac{x-8}{x-2}$.

38. $3=\dfrac{(x+b-c)(x-b+c)}{(b+c+x)(b+c-x)}$.

PROBLEMS

1. Find a number such that if it is added to each term of the fraction $\frac{3}{8}$ and subtracted from each term of the fraction $\frac{13}{24}$ the results will be equal.

2. Make and solve a general problem of which 1 is a special case.

3. Three times one of two numbers is 4 times the other. If the sum of their squares is divided by the sum of the numbers, the quotient is $42\frac{6}{7}$ less than that obtained by dividing the sum of the squares by the difference of the numbers. Find the numbers.

4. The sum of two numbers less 2, divided by their difference, is 4, and the sum of their cubes divided by the difference of their squares is $1\frac{2}{5}$ times their sum. Find the numbers.

5. The circumference of the rear wheel of a carriage is 4 feet greater than that of the front wheel. In running one mile the front wheel makes 110 revolutions more than the rear wheel. Find the circumference of each wheel.

6. State and solve a general problem of which 5 is a special case, using b feet instead of one mile, letting the other numbers remain as they are in problem 5.

7. In going one mile the front wheel of a carriage makes 88 revolutions more than the rear wheel. If one foot is added to the circumference of the rear wheel, and 3 feet to that of the front wheel, the latter will make 22 revolutions more than the former. Find the circumference of each wheel.

8. State and solve a general problem of which 7 is a special case, using a instead of 88, letting the other numbers remain as they are.

9. The circumference of the front wheel of a carriage is a feet, and that of the rear wheel b feet. In going a certain distance the front wheel makes n revolutions more than the rear wheel. Find the distance.

10. State and solve a problem which is a special case of problem 9, using the formula just obtained.

11. There is a number consisting of two digits whose sum, divided by their difference, is 4. The number divided by the sum of its digits is equal to twice the digit in units' place plus $\frac{1}{8}$ of the digit in tens' place. Find the number.

12. There is a fraction such that if 3 is added to each of its terms, the result is $\frac{4}{5}$, and if 3 is subtracted from each of its terms, the result is $\frac{1}{2}$. Find the fraction.

13. State and solve a general problem of which 12 is a special case.

14. A and B working together can do a piece of work in 6 days. A can do it alone in 5 days less than B. How long will it require each when working alone?

15. State and solve a general problem of which **14** is a special case.

16. On her second westward trip the *Mauritania* traveled 625 knots in a certain time. If her speed had been 5 knots less per hour, it would have required $6\frac{1}{4}$ hours longer to cover the same distance. Find her speed per hour.

17. By increasing the speed a miles per hour, it requires b hours less to go c miles. Find the original speed. Show how problem 16 may be solved by means of the formula thus obtained.

18. A train is to run d miles in a hours. After going c miles a dispatch is received requiring the train to reach its destination b hours earlier. What must be the speed of the train for the remainder of the journey?

19. A man can row a miles down stream and return in b hours. If his rate up stream is c miles per hour less than down stream, find the rate of the current, and the rate of the boat in still water.

20. State and solve a special case of problem 19.

21. A can do a piece of work in a days, B can do it in b days, and C in c days. How long will it require all working together to do it?

22. Three partners, A, B, and C, are to divide a profit of p dollars. A had put in a dollars for m months, B had put in b dollars for n months, and C c dollars for t months. What share of the profit does each get?

23. State and solve a problem which is a special case of the preceding problem.

CHAPTER IX

RATIO, VARIATION, AND PROPORTION

RATIO AND VARIATION

166. In many important applications fractions are called **ratios.**

E.g. $\frac{3}{5}$ is called the ratio of 3 to 5 and is sometimes written $3:5$.

It is to be understood that a ratio is the *quotient of two numbers* and hence is itself a *number.* We sometimes speak of the ratio of two magnitudes of the same kind, meaning thereby that these magnitudes are expressed in terms of a common unit and a ratio formed from the resulting *numbers.*

E.g. If, on measuring, the heights of two trees are found to be 25 feet and 35 feet respectively, we say the *ratio of their heights* is $\frac{25}{35}$ or $\frac{5}{7}$.

167. Two magnitudes are said to be **incommensurable** if there is no common unit of measure which is contained exactly an integral number of times in each.

E.g. If a and d are the lengths of the side and the diagonal of a square, then $d^2 = a^2 + a^2$, § 151, E. C. Hence, $\frac{a^2}{d^2} = \frac{1}{2}$ or $\frac{a}{d} = \frac{1}{\sqrt{2}}$. But since $\sqrt{2}$ is neither an *integer* nor a *fraction* (§ 108), it follows that a and d have no common measure, that is, they are *incommensurable.*

168. In many problems, especially in Physics, magnitudes are considered which are constantly changing. Number expressions representing such magnitudes are called **variables,** while those which represent fixed magnitudes are **constants.**

E.g. Suppose a body is moving at a uniform rate of 5 ft. per second. If t is the *number* of seconds from the time of starting and s the *number* of feet passed over, then s and t are *variables.*

The variables s and t, in case of uniform motion, have a *fixed ratio;* namely, in this example, $s:t=5$ for every pair of corresponding values of s and t throughout the period of motion.

169. When two variables are so related that for all pairs of corresponding values, their *ratio remains constant,* then each one is said to **vary directly as the other.**

E.g. If $s:t=k$ (a constant) then s varies directly as t, and t varies directly as s.

Variation is sometimes indicated by the symbol $\propto$. Thus $s \propto t$ means s *varies as* t, i.e. $\frac{s}{t}=k$ or $s=kt$.

170. When two variables are so related that for all pairs of corresponding values their *product remains constant,* than each one is said to **vary inversely as the other.**

E.g. Consider a rectangle whose area is A and whose base and altitude are b and h respectively. Then, $A=h\cdot b$.

If now the base is multiplied by 2, 3, 4, etc., while the altitude is divided by 2, 3, 4, etc., then the area will remain constant. Hence, b and h may both *vary* while A remains *constant.*

The relation $b\cdot h=A$ may be written $b=A\cdot\frac{1}{h}$ or $h=A\cdot\frac{1}{b}$. It may also be written $b:\frac{1}{h}=A$ or $h:\frac{1}{b}=A$, so that the ratio of either b or h to the *reciprocal* of the other is the constant A. For this reason one is said to *vary inversely as the other.*

171. If $y=kx^2$, k being constant and x and y variables, then y varies **directly as x^2**. If $y=\frac{k}{x^2}$, then y varies **inversely as x^2**. If $y=k\cdot wx$, then y **varies jointly as w and x**. If $y\propto wx$, then $y \propto w$ if x is constant and $y\propto x$ if w is constant. If $y=k\cdot\frac{w}{x}$, then y **varies directly as w and inversely as x**.

Example. The resistance offered by a wire to an electric current varies directly as its length and inversely as the area of its cross section.

If a wire $\frac{1}{8}$ in. in diameter has a resistance of r units per mile, find the resistance of a wire $\frac{1}{4}$ in. in diameter and 3 miles long.

Solution. Let R represent the resistance of a wire of length l and cross-section area $s = \pi \cdot (\text{radius})^2$. Then $R = k \cdot \frac{l}{s}$ where k is some constant. Since $R = r$ when $l = 1$ and $s = \pi(\frac{1}{16})^2$, we have

$$r = k \cdot \frac{1}{\frac{\pi}{256}} \text{ or } k = \frac{\pi r}{256}$$

Hence, when $l = 3$ and $s = \pi(\frac{1}{8})^2$, we have,

$$R = \frac{\pi r}{256} \cdot \frac{3}{\frac{\pi}{64}} = \frac{3}{4} r.$$

That is, the resistance of three miles of the second wire is $\frac{3}{4}$ the resistance *per mile* of the first wire.

PROBLEMS

1. If $z \propto w$, and if $z = 27$ when $w = 3$, find the value of z when $w = 4\frac{1}{3}$.

2. If z varies jointly as w and x, and if $z = 24$ when $w = 2$ and $x = 3$, find z when $w = 3\frac{1}{2}$ and $x = 7$.

3. If z varies inversely as w, and if $z = 11$ when $w = 3$, find z when $w = 66$.

4. If z varies directly as w and inversely as x, and if $z = 28$ when $w = 14$ and $x = 2$, find z when $w = 42$ and $x = 3$.

5. If z varies inversely as the square of w, and if $z = 3$ when $w = 2$, find z when $w = 6$.

6. If q varies directly as m and inversely as the square of d, and $q = 30$ when $m = 1$ and $d = \frac{6}{100}$, find q when $m = 3$ and $d = 600$.

7. If $y^2 \propto x^3$, and if $y = 16$ when $x = 4$, find y when $x = 9$.

8. The weight of a triangle cut from a steel plate of uniform thickness varies jointly as its base and altitude. Find the base when the altitude is 4 and the weight 72, if it is known that the weight is 60 when the altitude is 5 and base 6.

9. The weight of a circular piece of steel cut from a sheet of uniform thickness varies as the square of its radius. Find the weight of a piece whose radius is 13 ft., if a piece of radius 7 feet weighs 196 pounds.

10. If a body starts falling from rest, its velocity varies directly as the number of seconds during which it has fallen. If the velocity at the end of 3 seconds is 96.6 feet per second, find its velocity at the end of 7 seconds; of ten seconds.

11. If a body starts falling from rest, the total distance fallen varies directly as the square of the time during which it has fallen. If in 2 seconds it falls 64.4 feet, how far will it fall in 5 seconds? In 9 seconds?

12. The number of vibrations per second of a pendulum varies inversely as the square root of the length. If a pendulum 39.1 inches long vibrates once in each second, how long is a pendulum which vibrates 3 times in each second?

13. Illuminating gas in cities is forced through the pipes by subjecting it to pressure in the storage tanks. It is found that the volume of gas varies inversely as the pressure. A certain body of gas occupies 49,000 cu. ft. when under a pressure of 2 pounds per square inch. What space would it occupy under a pressure of $2\frac{4}{5}$ pounds per square inch?

14. The amount of heat received from a stove varies inversely as the square of the distance from it. A person sitting 15 feet from the stove moves up to 5 feet from it. How much will this increase the amount of heat received?

15. The weights of bodies of the same shape and of the same material vary as the cubes of corresponding dimensions. If a ball $3\frac{1}{4}$ inches in diameter weighs 14 oz., how much will a ball of the same material weigh whose diameter is $3\frac{1}{2}$ inches?

16. On the principle of problem 15, if a man 5 feet 9 inches tall weighs 165 pounds, what should be the weight of a man of similar build 6 feet tall?

PROPORTION

172. Definitions. The four numbers a, b, c, d are said to be **proportional** or to **form a proportion** if the ratio of a to b is equal to the ratio of c to d. That is, if $\frac{a}{b}=\frac{c}{d}$. This is also sometimes written $a:b::c:d$, and is read a is to b as c is to d.

The four numbers are called the **terms** of the proportion; the first and fourth are the **extremes**; the second and third the **means** of the proportion. The first and third are the **antecedents** of the ratios, the second and fourth the **consequents.**

If a, b, c, x are proportional, x is called the **fourth proportional** to a, b, c. If a, x, x, b are proportional, x is called a **mean proportional** to a and b, and b a **third proportional** to a and x.

173. If four numbers are proportional when taken in a *given order*, there are other orders in which they are also proportional.

E.g. If a, b, c, d are proportional in this order, they are also proportional in the following orders: a, c, b, d; b, a, d, c; b, d, a, c; c, a, d, b; c, d, a, b; d, c, b, a; and d, b, c, a.

Ex. 1. Write in the form of an equation the proportion corresponding to each set of four numbers given above, and show how each may be derived from $\frac{a}{b}=\frac{c}{d}$. See § 196, E. C.

Show first how to derive $\frac{a}{c}=\frac{b}{d}$ (1), and then $\frac{b}{a}=\frac{d}{c}$ (2).

Derive also $\frac{a+b}{a}=\frac{c+d}{c}$ (3), and $\frac{a-b}{a}=\frac{c-d}{c}$ (4).

In (1) the original proportion is said to be taken by **alternation**, and in (2) by **inversion**; in (3) by **composition**, and in (4) by **division.**

Ex. 2. From $\frac{a}{b}=\frac{c}{d}$ and (1), (2), (3), (4) obtain the following. See pp. 279–281, E. C.

$$\frac{a \pm b}{b}=\frac{c \pm d}{d}, \qquad \frac{a \pm b}{c \pm d}=\frac{a}{c}, \qquad \frac{a \pm c}{b \pm d}=\frac{a}{b}, \qquad \frac{a \pm b}{c + d}=\frac{a-b}{c \mp d}.$$

When the double sign occurs, the *upper* signs are to be read together and the *lower* signs together.

[illegible] V

EXPONENTS AND RADICALS

[illegible] AND NEGATIVE EXPONENTS

[illegible] the word *exponent* [illegible] number [illegible]

[illegible] indicate the number of [illegible]

[illegible] and negative [illegible] pos- [illegible]

[illegible] are

[illegible] § 43

[illegible] § 46

[illegible] § 115

[illegible] § 116

[illegible] § 117

[illegible]

[illegible]

See § 124

Similarly, from $\left(b^{\frac{1}{s}}\right)^r = b^{\frac{1}{s}} \cdot b^{\frac{1}{s}} \cdots$ to r factors, $= b^{\frac{r}{s}}$,

we show that $$b^{\frac{r}{s}} = \left(b^{\frac{1}{s}}\right)^r = (\sqrt[s]{b})^r.$$

Hence, $$\sqrt[s]{b^r} = (\sqrt[s]{b})^r. \qquad \text{See § 119}$$

Thus a positive fractional exponent means *a root of a power or a power of a root, the numerator indicating the power and the denominator indicating the root.*

E.g. $a^{\frac{2}{3}} = \sqrt[3]{a^2} = (\sqrt[3]{a})^2$; $8^{\frac{2}{3}} = \sqrt[3]{64} = 4$, or $(\sqrt[3]{8})^2 = 2^2 = 4$.

177. Assuming Law I to hold also for negative exponents, and letting t be a positive number, integral or fractional, we determine as follows the meaning of b^{-t} (*read b exponent negative t*).

By Law I, $$b^t \cdot b^{-t} = b^0 = 1. \qquad \text{§ 16}$$

Therefore, $$b^{-t} = \frac{1}{b^t}. \qquad \text{§ 11}$$

Hence a number with a negative exponent means *the same as the reciprocal of the number with a positive exponent of the same absolute value.*

E.g. $a^{-2} = \dfrac{1}{a^2}$; $4^{-\frac{3}{2}} = \dfrac{1}{4^{\frac{3}{2}}} = \dfrac{1}{2^3} = \dfrac{1}{8}$.

178. It thus appears that fractional and negative exponents simply provide *new ways of indicating operations already well known.* Sometimes one notation is more convenient and sometimes the other.

Fractional and negative exponents are also called *powers.*

E.g. $x^{\frac{2}{3}}$ may be read *x to the* $\frac{2}{3}$ *power*, and x^{-4} may be read *x to the −4th power.*

The limitations as to principal roots and the sign of the base, imposed in theorems on powers and roots in Chapter VI, necessarily apply to the corresponding theorems in this chapter. See §§ 114–122.

CHAPTER X

EXPONENTS AND RADICALS

FRACTIONAL AND NEGATIVE EXPONENTS

174. The meaning heretofore attached to the word *exponent* cannot apply to a fractional or negative number.

E.g. Such an exponent as $\frac{2}{3}$ or -5 cannot indicate the *number of times* a base is used as a factor.

It is possible, however, to interpret fractional and negative exponents in such a way *that the laws of operations which govern positive integral exponents shall apply to these also.*

175. The laws for positive integral exponents are:

I. $a^m \cdot a^n = a^{m+n}$. § 43

II. $a^m \div a^n = a^{m-n}$. § 46

III. $(a^m)^n = a^{mn}$. § 115

IV. $(a^m \cdot b^n)^p = a^{mp}b^{np}$. § 116

V. $(a^m \div s^n)^p = a^{mp} \div s^{np}$. § 117

176. Assuming Law I to hold for positive fractional exponents and letting r and s be positive integers, we determine as follows the meaning of $b^{\frac{r}{s}}$ (read *b exponent r divided by s*).

By definition, $\left(b^{\frac{r}{s}}\right)^s = b^{\frac{r}{s}} \cdot b^{\frac{r}{s}} \cdots$ to s factors,

which by Law I $= b^{\frac{r}{s}+\frac{r}{s}+\cdots \text{ to } s \text{ terms}} = b^{\frac{r}{s}\cdot s} = b^r$.

Hence, $b^{\frac{r}{s}}$ is one of the s equal factors of b^r.

That is, $b^{\frac{r}{s}} = \sqrt[s]{b^r}$, and in particular $b^{\frac{1}{s}} = \sqrt[s]{b}$. See § 114

Similarly, from $\left(b^{\frac{1}{s}}\right)^r = b^{\frac{1}{s}} \cdot b^{\frac{1}{s}} \ldots$ to r factors, $= b^{\frac{r}{s}}$,

we show that $\qquad b^{\frac{r}{s}} = \left(b^{\frac{1}{s}}\right)^r = (\sqrt[s]{b})^r.$

Hence, $\qquad b^{\frac{r}{s}} = \sqrt[s]{b^r} = (\sqrt[s]{b})^r.$ See § 119

Thus a positive fractional exponent means *a root of a power or a power of a root, the numerator indicating the power and the denominator indicating the root.*

E.g. $a^{\frac{2}{3}} = \sqrt[3]{a^2} = (\sqrt[3]{a})^2$; $8^{\frac{2}{3}} = \sqrt[3]{64} = 4$, or $(\sqrt[3]{8})^2 = 2^2 = 4$.

177. Assuming Law I to hold also for negative exponents, and letting t be a positive number, integral or fractional, we determine as follows the meaning of b^{-t} (*read b exponent negative t*).

By Law I, $\qquad b^t \cdot b^{-t} = b^0 = 1.$ § 46

Therefore, $\qquad b^{-t} = \dfrac{1}{b^t}.$ § 11

Hence a number with a negative exponent means *the same as the reciprocal of the number with a positive exponent of the same absolute value.*

E.g. $a^{-2} = \dfrac{1}{a^2}$; $\qquad 4^{-\frac{3}{2}} = \dfrac{1}{4^{\frac{3}{2}}} = \dfrac{1}{2^3} = \dfrac{1}{8}.$

178. It thus appears that fractional and negative exponents simply provide *new ways of indicating operations already well known.* Sometimes one notation is more convenient and sometimes the other.

Fractional and negative exponents are also called *powers.*

E.g. $x^{\frac{2}{3}}$ may be read *x to the* $\frac{2}{3}$ *power*, and x^{-4} may be read *x to the* $-$ *4th power.*

The limitations as to principal roots and the sign of the base, imposed in theorems on powers and roots in Chapter VI, necessarily apply to the corresponding theorems in this chapter. See §§ 114–122.

In any algebraic expression, radical signs may now be replaced by fractional exponents, or fractional exponents by radical signs.

In a fraction, any *factor* may be changed from numerator to denominator, or from denominator to numerator, by changing the sign of its exponent.

Ex. 1. $\sqrt[3]{x^2} + 3\sqrt[5]{x^3} \cdot \sqrt{y} + 5\sqrt[7]{x^4}\sqrt[3]{y^2} \equiv x^{\frac{2}{3}} + 3x^{\frac{3}{5}}y^{\frac{1}{2}} + 5x^{\frac{4}{7}}y^{\frac{2}{3}}.$

Ex. 2. $\dfrac{ab}{x^2} = abx^{-2}$, since $abx^{-2} = ab \cdot \dfrac{1}{x^2} = \dfrac{ab}{x^2}.$

Ex. 3. $ab^{-3}c^2 = ac^2 \cdot \dfrac{1}{b^3} = \dfrac{ac^2}{b^3}.$

Ex. 4. $32^{-\frac{4}{5}} = \dfrac{1}{32^{\frac{4}{5}}} = \dfrac{1}{(\sqrt[5]{32})^4} = \dfrac{1}{2^4} = \dfrac{1}{16}.$

EXERCISES

(*a*) In the expressions containing radicals on p. 154, replace these by fractional exponents.

(*b*) Replace all positive fractional exponents on this page by radicals.

(*c*) Change all expressions containing negative exponents to equivalent expressions having only positive exponents.

179. Fractional and negative exponents have been defined so as to conform to Law I, §§ 176, 177. We now show that when so defined they also conform to Laws II, III, IV, and V.

To verify Law II. Since by Law I, $a^{m-n} \cdot a^n = a^m$, for m and n integral or fractional, positive or negative, it follows by § 11 that $a^m \div a^n = a^{m-n}$ for all rational exponents.

To verify Law III. Let r and s be positive integers, and let k be any positive or negative integer or fraction. Then we have:

(1) $(a^k)^{\frac{r}{s}} = \sqrt[s]{(a^k)^r} = \sqrt[s]{a^{kr}} = a^{\frac{rk}{s}} = a^{\frac{r}{s}\cdot k}$, by §§ 176, 115.

(2) $(a^k)^{-\frac{r}{s}} = \dfrac{1}{(a^k)^{\frac{r}{s}}} = \dfrac{1}{a^{\frac{r}{s}\cdot k}} = a^{-\frac{r}{s}\cdot k}$, by § 177 and (1).

Hence $(a^k)^n = a^{nk}$ for all rational values of n and k.

To verify Law IV. Let m and n be positive or negative integers or fractions, and let r and s be positive integers, then we have

(1) $(a^m b^n)^{\frac{r}{s}} = \sqrt[s]{(a^m b^n)^r} = \sqrt[s]{a^{mr} b^{nr}}$, by §§ 176, 115,

$= \sqrt[s]{a^{mr}} \cdot \sqrt[s]{b^{nr}} = a^{\frac{r}{s}\cdot m} \cdot b^{\frac{r}{s}\cdot n}$, by §§ 120, 176.

(2) $(a^m b^n)^{-\frac{r}{s}} = \dfrac{1}{(a^m b^n)^{\frac{r}{s}}} = \dfrac{1}{a^{\frac{r}{s}\cdot m} \cdot b^{\frac{r}{s}\cdot n}} = a^{-\frac{r}{s}\cdot m} \cdot b^{-\frac{r}{s}\cdot n}$, by § 177 and (1).

Hence $(a^m b^n)^p = a^{pm} b^{pn}$ for all rational values of m, n, and p.

To verify Law V. We have $\left(\dfrac{a^m}{b^n}\right)^p = \dfrac{a^{mp}}{b^{np}}$ for all rational values of m, n, and p, since

$$\left(\frac{a^m}{b^n}\right)^p = (a^m \cdot b^{-n})^p = a^{mp} \cdot b^{-np} = \frac{a^{mp}}{b^{np}}, \text{ § 177 and Law IV.}$$

180. From Laws III, IV, and V, it follows that any monomial is affected with any exponent by multiplying the exponent of each factor of the monomial by the given exponent.

Ex. 1. $(a^{\frac{1}{2}} b^{-2} c^3)^{-\frac{2}{3}} = a^{-\frac{2}{3}\cdot\frac{1}{2}} b^{-\frac{2}{3}\cdot -2} c^{-\frac{2}{3}\cdot 3} = a^{-\frac{1}{3}} b^{\frac{4}{3}} c^{-2}$.

Ex. 2. $\left(\dfrac{3a^2x^6}{by^4}\right)^{-\frac{1}{2}} = \dfrac{3^{-\frac{1}{2}} a^{-1} x^{-3}}{b^{-\frac{1}{2}} y^{-2}} = \dfrac{b^{\frac{1}{2}} y^2}{3^{\frac{1}{2}} a x^3}$.

Ex. 3. $\left(\dfrac{8x^9}{27y^6}\right)^{-\frac{1}{3}} = \left(\dfrac{27y^6}{8x^9}\right)^{\frac{1}{3}} = \dfrac{27^{\frac{1}{3}} y^2}{8^{\frac{1}{3}} x^3} = \dfrac{3y^2}{2x^3}$.

EXERCISES

Perform the operations indicated by the exponents in each of the following, writing the results without negative exponents and in as simple form as possible:

1. $(\frac{16}{25})^{-\frac{1}{2}}$.	**5.** $(x^{-\frac{2}{3}} y^{\frac{4}{3}})^{-\frac{3}{4}}$.	**9.** $(\frac{27}{8})^{-\frac{2}{3}}$.	**13.** $(.0009)^{\frac{3}{2}}$.
2. $(\frac{27}{64})^{-\frac{1}{3}}$.	**6.** $25^{\frac{3}{2}}$.	**10.** $(\frac{8}{27})^{\frac{2}{3}}$.	**14.** $(.027)^{\frac{1}{3}}$.
3. $(\frac{25}{36})^{\frac{3}{2}}$.	**7.** $25^{-\frac{3}{2}}$.	**11.** $(0.25)^{\frac{1}{2}}$.	**15.** $(32a^{-5}b^{10})^{\frac{1}{5}}$.
4. $(27a^{-9})^{\frac{1}{3}}$.	**8.** 25^0.	**12.** $(0.25)^{-\frac{1}{2}}$.	**16.** $8^{\frac{1}{3}} \cdot 4^{-\frac{1}{2}}$.

17. $\left(\dfrac{a^{-8}}{16}\right)^{-\frac{1}{4}}$.

19. $\left(\frac{1}{32}\right)^{-\frac{1}{5}}\left(\frac{1}{81}\right)^{-\frac{3}{4}}$.

21. $\left(-\frac{243}{32}\right)^{\frac{1}{5}} \div \left(\frac{16}{81}\right)^{-\frac{1}{4}}$.

18. $(27\,x^6y^{-3}z^{-1})^{-\frac{1}{3}}$.

20. $\sqrt[3]{\frac{512}{729}} \cdot \left(\frac{256}{729}\right)^{-\frac{1}{2}}$.

22. $\left(\dfrac{x^3y^{-4}}{x^{-2}y}\right)^3\left(\dfrac{x^{-3}y^2}{xy^{-1}}\right)^5$.

23. $\sqrt[4]{81\,a^{-4}b^8}\,(-27\,a^3b^{-6})^{-\frac{1}{3}}$.

26. $\sqrt{16\,a^{-4}b^{-6}} \cdot \sqrt[3]{8\,a\,b^{-6}}$.

24. $\left(\dfrac{m^{-1}n}{m^{\frac{1}{2}}n^{-\frac{2}{3}}}\right)^{-2} \div \left(\dfrac{m^{-3}}{n^{-1}}\right)^{-\frac{1}{3}}$.

27. $(-2^{-2}a^{-3}b^{-6})^{-\frac{1}{3}}(-2^{-\frac{1}{3}}a^{-\frac{1}{2}}b^{-1})^2$.

25. $\left(\dfrac{a^3b^{-2}}{a^{-2}b^3}\right)^{\frac{1}{3}} \div \left(\dfrac{a^3b^{-3}}{a^{-3}b^3}\right)^{-1}$.

28. $\left(\dfrac{81\,r^{-16}s^4t}{625\,r^4s^8t}\right)^{\frac{1}{4}}\left(\dfrac{9\,r^2s^{-4}t}{25\,r^{10}s^6t}\right)^{-\frac{1}{2}}$.

29. Prove Law III in detail for the following cases:

$$(1)\ (a^{\frac{m}{n}})^{-\frac{r}{s}},\quad (2)\ (a^{-\frac{m}{n}})^{\frac{r}{s}},\quad (3)\ (a^{-\frac{m}{n}})^{-\frac{r}{s}}.$$

30. Prove Law IV in detail for the following cases:

$$(1)\ (a^{-k}b^{-l})^{\frac{r}{s}},\quad (2)\ (a^{-k}b^{-l})^{-\frac{r}{s}}.$$

Multiply:

31. $x^{-2}+x^{-1}y^{-1}+y^{-2}$ by $x^{-1}-y^{-1}$.

32. $x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$ by $x^{\frac{1}{3}}+y^{\frac{1}{3}}$.

33. $x^{\frac{4}{5}}+x^{\frac{3}{5}}y^{\frac{1}{5}}+x^{\frac{2}{5}}y^{\frac{2}{5}}+x^{\frac{1}{5}}y^{\frac{3}{5}}+y^{\frac{4}{5}}$ by $x^{\frac{1}{5}}-y^{\frac{1}{5}}$.

34. $\sqrt[4]{a^3}+\sqrt[5]{b^2}$ by $\sqrt[4]{a^3}-\sqrt[5]{b^2}$.

35. $x^{\frac{4}{3}}+x^{\frac{2}{3}}y^{\frac{2}{3}}+y^{\frac{4}{3}}$ by $x^{\frac{2}{3}}-y^{\frac{2}{3}}$.

36. $x-3\,x^{\frac{2}{3}}y^{-\frac{1}{2}}+3\,x^{\frac{1}{3}}y^{-1}-y^{-\frac{3}{2}}$ by $x^{\frac{2}{3}}-2\,x^{\frac{1}{3}}y^{-\frac{1}{2}}+y^{-1}$.

37. $x^{\frac{3}{2}}+xy^{\frac{3}{4}}+x^{\frac{1}{2}}y^{\frac{3}{2}}+y^{\frac{9}{4}}$ by $x^{\frac{1}{2}}-y^{\frac{3}{4}}$.

Divide:

38. $x^2-x^{\frac{11}{6}}y+x^{\frac{3}{2}}y-x^{\frac{1}{2}}y^{\frac{1}{3}}+x^{\frac{1}{3}}y^{\frac{4}{3}}-y^{\frac{4}{3}}$ by $x^{\frac{1}{2}}-x^{\frac{1}{3}}y+y$.

39. $3\,a^{\frac{7}{4}}-ab^{\frac{2}{3}}+4\,ab^2-3\,a^{\frac{3}{4}}b+b^{\frac{5}{3}}-4\,b^3$ by $3\,a^{\frac{3}{4}}-b^{\frac{2}{3}}+4\,b^2$.

40. $x^2-3\,x^{\frac{5}{3}}+6\,x^{\frac{4}{3}}-7\,x+6\,x^{\frac{2}{3}}-3\,x^{\frac{1}{3}}+1$ by $x^{\frac{2}{3}}-x^{\frac{1}{3}}+1$.

41. $4\,x^{\frac{5}{4}}b^{-2}-17\,x^{\frac{1}{4}}b^2+16\,x^{-\frac{3}{4}}b^6$ by $2\,x^{\frac{1}{2}}-b^2-4\,x^{-\frac{1}{2}}b^4$.

Find the square root of:

42. $4x^2 - 4xy^{\frac{1}{3}} + 4xz^{-\frac{1}{2}} + y^{\frac{2}{3}} - 2y^{\frac{1}{3}}z^{-\frac{1}{2}} + z^{-1}$.

43. $a^{-\frac{2}{3}} - 2a^{\frac{1}{3}}b^{\frac{2}{3}} + b^{\frac{4}{3}} + 2a^{-\frac{1}{3}}c^2 + c^4 - 2b^{\frac{2}{3}}c^2$.

44. $b^{-\frac{1}{2}} - 2b^{-\frac{1}{4}}c^{\frac{1}{3}} + c^{\frac{2}{3}} + 2b^{-\frac{1}{4}}d^{\frac{1}{5}} + 2b^{-\frac{1}{4}}e^{-\frac{1}{2}} - 2c^{\frac{1}{3}}d^{\frac{1}{5}} + d^{\frac{2}{5}}$
$+ 2d^{\frac{1}{5}}e^{-\frac{1}{2}} - 2c^{\frac{1}{3}}e^{-\frac{1}{2}} + e^{-1}$.

Find the cube root of: **45.** $\frac{1}{8}a^3 - \frac{3}{2}a^2b^{\frac{1}{2}} + 6ab - 8b^{\frac{3}{2}}$.

46. $a^6 - 3a^5 + 5a^3 - 3a - 1$. **47.** $a^{-1} + 3a^{-\frac{2}{3}}b^{\frac{2}{5}} + 3a^{-\frac{1}{3}}b^{\frac{4}{5}} + b^{\frac{6}{5}}$.

REDUCTION OF RADICAL EXPRESSIONS

181. An expression containing a root indicated by the radical sign or by a fractional exponent is called a **radical expression.** The expression whose root is indicated is the **radicand.**

E.g. $\sqrt[3]{5}$ and $(1 + x)^{\frac{2}{3}}$ are radical expressions. In each case the **index** of the radical is 3.

The reduction of a radical expression consists in *changing its form without changing its value.*

Each reduction is based upon one or more of the Laws I to V, § 175, as extended in § 179.

182. To remove a factor from the radicand. This reduction is possible only when the radicand contains a factor which is a perfect power of the degree indicated by the index of the root, as shown in the following examples:

Ex. 1. $\sqrt{72} = \sqrt{36 \cdot 2} = \sqrt{36}\sqrt{2} = 6\sqrt{2}$.

Ex. 2. $(a^3x^2y^6)^{\frac{1}{3}} = (a^3y^6 \cdot x^2)^{\frac{1}{3}} = (a^3y^6)^{\frac{1}{3}} \cdot (x^2)^{\frac{1}{3}} = ay^2x^{\frac{2}{3}}$.

This reduction involves Law IV, and may be written in symbols thus:

$$\sqrt[r]{x^{kr}y} = \sqrt[r]{x^{kr}}\,\sqrt[r]{y} = x^k\sqrt[r]{y}.$$

EXERCISES

In the expressions on p. 154, remove factors from the radicands where possible.

In the case of negative fractional exponents, first reduce to equivalent expressions containing only positive exponents.

183. To introduce a factor into the radicand. This process simply retraces the steps of the foregoing reduction, and hence also involves Law IV.

Ex. 1. $6\sqrt{2} = \sqrt{6^2}\cdot\sqrt{2} = \sqrt{36\cdot 2} = \sqrt{72}$. See § 112

Ex. 2. $ay^2x^{\frac{2}{3}} = \sqrt[3]{(ay^2)^3}\cdot\sqrt[3]{x^2} = \sqrt[3]{(ay^2)^3x^2} = \sqrt[3]{a^3y^6x^2}$.

Ex. 3. $x\sqrt[r]{y} = \sqrt[r]{x^r}\sqrt[r]{y} = \sqrt[r]{x^ry}$.

EXERCISES

In the expressions on p. 154, introduce into the radicand any factor which appears as a coefficient of a radical.

184. To reduce a fractional radicand to the integral form. This reduction involves Law IV or Law V, and may always be accomplished.

Ex. 1. $\sqrt{\frac{3}{5}} = \sqrt{\frac{15}{25}} = \sqrt{\frac{1}{25}\cdot 15} = \frac{1}{5}\sqrt{15}$. Law IV

Ex. 2. $\left(\frac{a-b}{a+b}\right)^{\frac{1}{2}} = \left(\frac{a^2-b^2}{(a+b)^2}\right)^{\frac{1}{2}} = \frac{(a^2-b^2)^{\frac{1}{2}}}{[(a+b)^2]^{\frac{1}{2}}} = \frac{(a^2-b^2)^{\frac{1}{2}}}{a+b}$. Law V

Ex. 3. $\frac{\sqrt[3]{10}}{\sqrt[3]{5}} = \sqrt[3]{\frac{10}{5}} = \sqrt[3]{8} = 2$.

In symbols, we have $\sqrt[r]{\frac{a}{b}} = \sqrt[r]{\frac{ab^{r-1}}{b^r}} = \frac{\sqrt[r]{ab^{r-1}}}{\sqrt[r]{b^r}} = \frac{1}{b}\sqrt[r]{ab^{r-1}}$.

EXERCISES

In the expressions on p. 154, reduce each fractional radicand to the integral form.

In case negative exponents are involved, first reduce to equivalent expressions containing only positive exponents.

185. To reduce a radical to an equivalent radical of lower index. This reduction is effective when the radicand is a perfect power corresponding to *some factor of the index.*

Ex. 1. $\sqrt[6]{8} = 8^{\frac{1}{6}} = 8^{\left(\frac{1}{3 \cdot 2}\right)} = (8^{\frac{1}{3}})^{\frac{1}{2}} = 2^{\frac{1}{2}} = \sqrt{2}.$

Ex. 2. $\sqrt[4]{a^2 + 2\,ab + b^2} = \sqrt{\sqrt{a^2 + 2\,ab + b^2}} = \sqrt{a + b}.$

This reduction involves Law III as follows:

$$(x^{\frac{1}{r}})^{\frac{1}{s}} = (x^{\frac{1}{s}})^{\frac{1}{r}} = x^{\frac{1}{rs}},$$

from which we have

$$\sqrt[rs]{x} = \sqrt[r]{\sqrt[s]{x}} = \sqrt[s]{\sqrt[r]{x}}. \qquad \text{See § 114}$$

By this reduction a root whose index is a composite number is made to depend upon roots of lower degree.

E.g. A fourth root may be found by taking the square root twice; a sixth root, by taking a square root and then a cube root, etc. In the case of *literal* expressions this can be done only when the radicand is a perfect power of the degree indicated by the index of the root.

But when the radicand is expressed in Arabic figures, such roots may in any case be approximated as in § 127.

EXERCISES

In the expressions on p. 154, make the reduction above indicated where possible.

In the case of arithmetic radicands, approximate to two places of decimals such roots as can be made to depend upon square and cube roots.

186. To reduce a radical to an equivalent radical of higher index. This reduction is possible whenever the required index is a *multiple* of the given index. It is based on Law III as follows:

$$x^{\frac{r}{s}} = (x^{\frac{r}{s}})^{\frac{t}{t}} = x^{\frac{rt}{st}}. \qquad \text{See § 179}$$

Ex. 1. $\sqrt{a} = a^{\frac{1}{2}} = (a^{\frac{1}{2}})^{\frac{3}{3}} = a^{\frac{3}{6}} = \sqrt[6]{a^3}.$

Ex. 2. $\sqrt[3]{b} = b^{\frac{1}{3}} = b^{\frac{2}{6}} = \sqrt[6]{b^2}.$

Definition. Two radical expressions are said to be of the **same order** when their indicated roots have the *same index*.

By the above reduction two radicals of *different* orders may be changed to equivalent radicals of the *same* order, namely, a common multiple of the given indices.

E.g. $\sqrt{a}$ and $\sqrt[3]{b}$ in Exs. 1 and 2 above.

EXERCISES

In Exs. 3, 4, 6, 17, 18, 23, 28, 30, on p. 154, reduce the corresponding expressions in the first and second columns to equivalent radicals of the same order.

187. In general, radical expressions should be at once reduced so that the *order is as low* as possible and the *radicand is integral and as small* as possible. A radical is then said to be in its **simplest form.**

ADDITION AND SUBTRACTION OF RADICALS

188. Definition. Two or more radical expressions are said to be **similar** when they are of the same order and have the same radicands.

E.g. $3\sqrt{7}$ and $5\sqrt{7}$ are similar radicals as are also $a\sqrt[7]{x^4}$ and $bx^{\frac{4}{7}}$.

If two radicals can be reduced to similar radicals, they may be added or subtracted according to § 10.

Ex. 1. Find the sum of $\sqrt{8}$, $\sqrt{50}$, and $\sqrt{98}$.

By § 182, $\sqrt{8} = 2\sqrt{2}$, $\sqrt{50} = 5\sqrt{2}$, and $\sqrt{98} = 7\sqrt{2}$.

Hence $\sqrt{8} + \sqrt{50} + \sqrt{98} = 2\sqrt{2} + 5\sqrt{2} + 7\sqrt{2} = 14\sqrt{2}$.

Ex. 2. Simplify $\sqrt{\frac{1}{5}} - \sqrt{20} + \sqrt{3\frac{1}{5}}$.

By § 184, $\sqrt{\frac{1}{5}} = \frac{1}{5}\sqrt{5}$, $\sqrt{20} = 2\sqrt{5}$, $\sqrt{3\frac{1}{5}} = \sqrt{\frac{16}{5}} = 4\sqrt{\frac{1}{5}} = \frac{4}{5}\sqrt{5}$.

Hence $\sqrt{\frac{1}{5}} - \sqrt{20} + \sqrt{3\frac{1}{5}} = \frac{1}{5}\sqrt{5} - 2\sqrt{5} + \frac{4}{5}\sqrt{5} = -\sqrt{5}$.

If two radicals cannot be reduced to equivalent similar radicals, their sum can only be indicated.

E.g. The sum of $\sqrt{2}$ and $\sqrt[3]{5}$ is $\sqrt{2}+\sqrt[3]{5}$.

Observe, however, that

$$\sqrt{10}+\sqrt{6}=\sqrt{2}\cdot\sqrt{5}+\sqrt{2}\cdot\sqrt{3}=\sqrt{2}\,(\sqrt{5}+\sqrt{3}).$$

EXERCISES

(*a*) In Exs. 1, 2, 5, 7, 8, 19, 20, 21, p. 154, reduce each pair so as to involve similar radicals and add them.

(*b*) Perform the following indicated operations:

1. $\sqrt{28}+3\sqrt{7}-2\sqrt{63}$.
2. $\sqrt[3]{24}-\sqrt[3]{81}-\sqrt[3]{\frac{3}{125}}$.
3. $\sqrt[5]{a^6}+\sqrt[5]{a^{11}}-\sqrt[5]{32a}$.
4. $2\sqrt{48}-3\sqrt{12}+3\sqrt{\frac{1}{3}}$.
5. $\sqrt{\frac{1}{7}}+\sqrt{63}+5\sqrt{7}$.
6. $\sqrt{99}-11\sqrt{\frac{1}{11}}+\sqrt{44}$.
7. $2\sqrt{\frac{4}{7}}+3\sqrt{\frac{9}{7}}+\sqrt{175}$.
8. $\sqrt[4]{\frac{9}{625}}+6\sqrt{\frac{1}{3}}-\sqrt{12}$.
9. $\sqrt[6]{9}+\sqrt[9]{27}+\sqrt[3]{-24}$.
10. $x^2+\sqrt{a^3+a^2b}-\sqrt{(a^2-b^2)(a-b)}$.

MULTIPLICATION OF RADICALS

189. Radicals of the *same order* are multiplied according to § 120 by multiplying the radicands. If they are not of the same order, they may be reduced to the same order according to § 186.

E.g. $\sqrt{a}\cdot\sqrt[3]{b}=a^{\frac{1}{2}}b^{\frac{1}{3}}=a^{\frac{3}{6}}b^{\frac{2}{6}}=\sqrt[6]{a^3}\cdot\sqrt[6]{b^2}=\sqrt[6]{a^3b^2}$.

In many cases this reduction is not desirable. Thus, $\sqrt[3]{x^2}\cdot\sqrt{y^3}$ is written $x^{\frac{2}{3}}y^{\frac{3}{2}}$ rather than $\sqrt[6]{x^4y^9}$.

Radicals are multiplied by adding exponents when they are reduced to the *same base* with fractional exponents, § 176.

E.g. $\sqrt[5]{x^2}\cdot\sqrt{x^3}=x^{\frac{2}{5}}\cdot x^{\frac{3}{2}}=x^{\frac{2}{5}+\frac{3}{2}}=x^{\frac{19}{10}}$.

190. The principles just enumerated are used in connection with § 10 in multiplying polynomials containing radicals.

Ex. 1.
$$\begin{array}{lll}
3\sqrt{2} & + 2\sqrt{5} & \\
2\sqrt{2} & - 3\sqrt{5} & \\
\hline
6\cdot 2 & + 4\sqrt{10} & \\
 & - 9\sqrt{10} & - 6\cdot 5 \\
\hline
12 & - 5\sqrt{10} & - 30
\end{array}$$

Ex. 2.
$$\begin{array}{lll}
3\sqrt{2} & + 2\sqrt{5} & \\
3\sqrt{2} & - 2\sqrt{5} & \\
\hline
9\cdot 2 & + 6\sqrt{10} & \\
 & - 6\sqrt{10} & - 4\cdot 5 \\
\hline
18 & & - 20
\end{array}$$

Hence $(2\sqrt{2} + 2\sqrt{5})(2\sqrt{2} - 3\sqrt{5}) = -18 - 5\sqrt{10}$,

and $(3\sqrt{2} + 2\sqrt{5})(3\sqrt{2} - 2\sqrt{5}) = 18 - 20 = -2$.

EXERCISES

(*a*) In Exs. 21 to 38, p. 154, find the products of the corresponding expressions in the two columns.

(*b*) Find the following products:

1. $(3 + \sqrt{11})(3 - \sqrt{11})$.
2. $(3\sqrt{2} + 4\sqrt{5})(4\sqrt{2} - 5\sqrt{5})$.
3. $(2 + \sqrt{3} + \sqrt{5})(3 + \sqrt{3} - \sqrt{5})$.
4. $(3\sqrt{2} - 2\sqrt{18} + 2\sqrt{7})(2\sqrt{2} - \sqrt{18} - \sqrt{7})$.
5. $(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})(a^2 + ab + b^2)$.
6. $(\sqrt{\sqrt{13} + 3})(\sqrt{\sqrt{13} - 3})$.
7. $(\sqrt{2 + 3\sqrt{5}})(\sqrt{2 + 3\sqrt{5}})$.
8. $(3a - 2\sqrt{x})(4a + 3\sqrt{x})$.
9. $(3\sqrt{3} + 2\sqrt{6} - 4\sqrt{8})(3\sqrt{3} - 2\sqrt{6} + 4\sqrt{8})$.
10. $(\sqrt{a} + \sqrt{b} - \sqrt{c})(\sqrt{a} - \sqrt{b} + \sqrt{c})$.

11. $(a - \sqrt{b} - \sqrt{c})(a + \sqrt{b} + \sqrt{c})$.

12. $(2\sqrt{\frac{2}{3}} + 3\sqrt{\frac{3}{6}} + 4\sqrt{\frac{3}{2}})(2\sqrt{\frac{2}{3}} - 5\sqrt{\frac{3}{6}})$.

13. $(\sqrt[3]{a^2} + \sqrt[3]{b^2})(\sqrt[6]{a^2} + \sqrt[3]{a^2b^2} + \sqrt[9]{b^3})$.

14. $(\sqrt[4]{x^3} - y^3)^3$.

DIVISION OF RADICALS

191. Radicals are divided in accordance with Laws II and V. That is, the exponents are *subtracted* when the *bases* are the *same*, and the bases are *divided* when the *exponents* are the *same*. See §§ 179, 121.

Ex. 1. $\sqrt[5]{x^2} \div \sqrt{x^3} = x^{\frac{2}{5}} \div x^{\frac{3}{2}} = x^{\frac{2}{5}-\frac{3}{2}} = x^{-\frac{11}{10}}$.

Ex. 2. $x^{\frac{2}{3}} \div y^{\frac{2}{3}} = \left(\frac{x}{y}\right)^{\frac{2}{3}} = (xy^{-1})^{\frac{2}{3}} = \sqrt[3]{x^2y^{-2}}$.

Ex. 3. $\sqrt{a} \div \sqrt[3]{b} = a^{\frac{3}{6}} \div b^{\frac{2}{6}} = \left(\frac{a^3}{b^2}\right)^{\frac{1}{6}} = \sqrt[6]{a^3b^{-2}}$.

EXERCISES

(*a*) In each of the Exs. 1 to 20, on p. 154, divide the expression in the first column by that in the second.

(*b*) Perform the following divisions:

1. $(\sqrt{a^3} + 2\sqrt{a^5} - 3\sqrt{a}) \div 6\sqrt{a}$.

2. $(\sqrt{a} + \sqrt[3]{b} - c) \div \sqrt{c}$.

3. $(2\sqrt[3]{9} + 3\sqrt[3]{12} - 4\sqrt[3]{15}) \div \sqrt[3]{3}$.

4. $(4\sqrt[5]{7} - 8\sqrt[5]{21} + 6\sqrt[5]{42}) \div 2\sqrt[5]{7}$.

EXERCISES

1. $3\sqrt{45}$, $2\sqrt{125}$.
2. $a^{\frac{2}{3}}$, $a^{\frac{5}{3}}$.
3. $x^{\frac{2}{5}}$, $x^{\frac{5}{3}}$.
4. $3\sqrt{x^2y}$, $2\sqrt[3]{x^4}$.
5. $d\sqrt[3]{a^2b^5}$, $c\sqrt[3]{a^5b^2}$.
6. $7\sqrt{(a+b)^3}$, $11\sqrt[3]{(a-b)^6}$.
7. $\sqrt[3]{a^4}$, $a^{\frac{7}{3}}$.
8. $n\sqrt{m^5}$, $m^{\frac{3}{2}}$.
9. $\sqrt{\frac{7}{2}}$, $\sqrt{\frac{8}{7}}$.
10. $\sqrt{\frac{11}{72}}$, $\sqrt{\frac{23}{54}}$.
11. $\sqrt{\frac{36}{27}}$, $\sqrt{\frac{13}{18}}$.
12. $\dfrac{1}{b^{-\frac{1}{2}}}$, $r\sqrt{b^3}$.
13. $a^{-2}b^3$, $\dfrac{a^3}{b^4}$.
14. $\dfrac{3}{m^{-\frac{2}{7}}}$, $\dfrac{n^{-\frac{1}{3}}m^{-1}}{4}$.
15. $\dfrac{3a^{-\frac{3}{2}}}{c^{-2}b^3}$, $\dfrac{2a^4b^{-2}}{c^3}$.
16. $\sqrt[3]{\frac{3}{4}}$, $\sqrt[3]{\frac{1}{4}}$.
17. $\sqrt[4]{\frac{3}{8}}$, $\sqrt[6]{\frac{2}{9}}$.
18. $\sqrt[9]{\frac{3}{64}}$, $\sqrt[6]{\frac{2}{7}}$.
19. $18^{\frac{1}{2}}$, $\sqrt{32}$.
20. $\sqrt{12}$, $48^{\frac{1}{2}}$.
21. $3(50)^{\frac{1}{2}}$, $4(72)^{\frac{1}{2}}$.
22. $a^{\frac{3}{4}}$, $a^{\frac{5}{3}}$.
23. $ax^{\frac{7}{2}}$, $bx^{\frac{3}{4}}$.
24. $a^{\frac{2}{3}}b^{\frac{7}{5}}$, $a^{\frac{3}{2}}b^{\frac{5}{7}}$.
25. $m^{\frac{3}{4}}n^{\frac{4}{5}}l^{\frac{7}{8}}$, $m^{\frac{3}{2}}n^{\frac{2}{3}}l^{\frac{4}{5}}$.
26. $5\sqrt[5]{a^6b^4c^9}$, $3\sqrt[5]{a^3b^7c^{13}}$.
27. $\sqrt[2]{a^3}\sqrt[4]{b^{10}}$, $\sqrt[3]{b^9}\sqrt[4]{a^6}$.
28. $8^{\frac{2}{3}}$, $16^{\frac{3}{2}}$.
29. $25^{-\frac{1}{2}}$, $125^{-\frac{4}{3}}$.
30. $9^{\frac{3}{2}}$, $8^{\frac{2}{3}}$.
31. $(\frac{1}{9})^{\frac{1}{3}}$, $(\frac{1}{27})^{-\frac{1}{2}}$.
32. $\dfrac{5x^3y^5}{m^{-3}n^{-5}}$, $\dfrac{x^3y^5z^2}{m^{-1}n^{-2}l^{-3}}$.
33. $\dfrac{3a^{-\frac{3}{2}}b^{\frac{3}{2}}}{4a^{-\frac{1}{2}}b^{-\frac{2}{3}}}$, $\dfrac{4a^{-\frac{5}{8}}b^{-\frac{3}{2}}}{5a^{-\frac{1}{2}}b^{-\frac{1}{3}}}$.
34. $\dfrac{c^{-\frac{1}{2}}d^{-\frac{1}{3}}}{d^{-\frac{1}{4}}c^{-\frac{1}{5}}}$, $\dfrac{d^{\frac{3}{4}}c^9}{d^{-\frac{1}{4}}c^{-1}}$.
35. $5(a+b)^{-\frac{3}{2}}$, $3(a+b)^{-\frac{2}{3}}$.
36. $(-64)^{\frac{5}{3}}$, $-64^{\frac{5}{6}}$.
37. $(\frac{1}{27})^{\frac{1}{3}}$, $(\frac{1}{16})^{-\frac{3}{4}}$.
38. $\dfrac{1}{b}\sqrt[3]{\dfrac{a^3}{b}}$, $\frac{1}{3}\sqrt[3]{\frac{27}{5}}$.
39. $\sqrt[3]{4a^3-12a^2b+12ab^2-4b^3}$.
40. $\sqrt{(3a-2b)(9a^2-4b^2)}$.

192. Rationalizing the divisor. In case division by a radical expression cannot be carried out as in the foregoing examples, it is desirable to *rationalize* the denominator when possible.

Ex. 1. $\dfrac{\sqrt{2}}{\sqrt{5}} = \dfrac{\sqrt{2}\cdot\sqrt{5}}{\sqrt{5}\cdot\sqrt{5}} = \dfrac{\sqrt{10}}{5}.$

Ex. 2. $\dfrac{\sqrt{a}}{\sqrt{a}-\sqrt{b}} = \dfrac{\sqrt{a}(\sqrt{a}+\sqrt{b})}{(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})} = \dfrac{a+\sqrt{ab}}{a-b}.$

Evidently this is always possible when the divisor is a *monomial* or *binomial* radical expression of the second order.

The number by which numerator and denominator are multiplied is called the *rationalizing factor.*

For a monomial divisor, $\sqrt{x}$, it is $\sqrt{x}$ itself. For a binomial divisor, $\sqrt{x}\pm\sqrt{y}$, it is the same binomial with the opposite sign, $\sqrt{x}\mp\sqrt{y}$.

EXERCISES

Reduce each of the following to equivalent fractions having a rational denominator.

1. $\dfrac{3}{2-\sqrt{5}}.$

2. $\dfrac{7}{\sqrt{5}+\sqrt{3}}.$

3. $\dfrac{\sqrt{27}}{\sqrt{3}+\sqrt{11}}.$

4. $\dfrac{2-\sqrt{7}}{2+\sqrt{7}}.$

5. $\dfrac{\sqrt{2}-\sqrt{3}}{\sqrt{2}+\sqrt{3}}.$

6. $\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}.$

7. $\dfrac{3\sqrt{3}-2\sqrt{2}}{3\sqrt{3}+2\sqrt{2}}.$

8. $\dfrac{\sqrt{a^2+1}-\sqrt{a^2-1}}{\sqrt{a^2-1}+\sqrt{a^2+1}}.$

9. $\dfrac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}.$

10. $\dfrac{\sqrt{a-b}-\sqrt{a+b}}{\sqrt{a-b}+\sqrt{a+b}}.$

193. In finding the value of such an expression as $\frac{\sqrt{7}+\sqrt{3}}{\sqrt{7}-\sqrt{3}}$, the approximation of *two* square roots and division by a decimal fraction would be required. But $\frac{\sqrt{7}+\sqrt{3}}{\sqrt{7}-\sqrt{3}}$ equals $\frac{11+2\sqrt{21}}{4}$ which requires only *one* root and division by the integer 4.

EXERCISES

Find the approximate values of the following expressions to three places of decimals.

1. $\dfrac{3\sqrt{5}+4\sqrt{3}}{\sqrt{5}-\sqrt{3}}$.

2. $\dfrac{\sqrt{7}}{\sqrt{7}-\sqrt{2}}$.

3. $\dfrac{4\sqrt{3}}{\sqrt{3}-\sqrt{2}}$.

4. $\dfrac{11\sqrt{5}-3\sqrt{3}}{2\sqrt{5}+\sqrt{3}}$.

5. $\dfrac{7\sqrt{5}+3\sqrt{8}}{2\sqrt{5}-3\sqrt{2}}$.

6. $\dfrac{5\sqrt{19}-3\sqrt{7}}{3\sqrt{7}-\sqrt{19}}$.

7. $\dfrac{3\sqrt{2}-\sqrt{5}}{\sqrt{5}-6\sqrt{2}}$.

8. $\dfrac{5\sqrt{6}-7\sqrt{13}}{3\sqrt{13}-7\sqrt{6}}$.

194. **Square root of a radical expression.** A radical expression of the second order is sometimes a perfect square, and its **square root** may be written by inspection.

E.g. The square of $\sqrt{a} \pm \sqrt{b}$ is $a+b \pm 2\sqrt{ab}$. Hence if a radical expression can be put into the form $x \pm 2\sqrt{y}$, where x is the *sum* of two numbers a and b whose *product* is y, then $\sqrt{a} \pm \sqrt{b}$ is the *square root* of $x+2\sqrt{y}$.

Example. Find the square root of $5+\sqrt{24}$.

Since $5+\sqrt{24}=5+2\sqrt{6}$, in which 5 is the *sum* of 2 and 3, and 6 is their *product*, we have $\sqrt{5+\sqrt{24}}=\sqrt{2}+\sqrt{3}$.

EXERCISES

Find the square root of each of the following:

1. $3 - 2\sqrt{2}$.
2. $7 + \sqrt{40}$.
3. $8 - \sqrt{60}$.
4. $7 + 4\sqrt{3}$.
5. $24 - 6\sqrt{7}$.
6. $28 + 3\sqrt{12}$.

195. Radical expressions involving imaginaries. According to the definition, § 112, $(\sqrt{-1})^2 = -1$. Hence, $(\sqrt{-1})^3 = (\sqrt{-1})^2\sqrt{-1} = -\sqrt{-1}$ and $(\sqrt{-1})^4 = (\sqrt{-1})^2(\sqrt{-1})^2 = (-1)(-1) = +1$.

The following examples illustrate operations with radical expressions containing imaginaries.

Ex. 1. $\sqrt{-4} + \sqrt{-9} = \sqrt{4}\sqrt{-1} + \sqrt{9}\sqrt{-1}$
$$= (2+3)\sqrt{-1} = 5\sqrt{-1}.$$

Ex. 2. $\sqrt{-4} \cdot \sqrt{-9} = \sqrt{4} \cdot \sqrt{9} \cdot (\sqrt{-1})^2 = -2 \cdot 3 = -6$.

Ex. 3. $\dfrac{\sqrt{-4}}{\sqrt{-9}} = \dfrac{\sqrt{4}\sqrt{-1}}{\sqrt{9}\sqrt{-1}} = \dfrac{\sqrt{4}}{\sqrt{9}} = \dfrac{2}{3}$.

Ex. 4. $\sqrt{-2} \cdot \sqrt{-3} \cdot \sqrt{-6} = \sqrt{2} \cdot \sqrt{3} \cdot \sqrt{6} \cdot (\sqrt{-1})^3$
$$= -\sqrt{36}\sqrt{-1} = -6\sqrt{-1}.$$

Ex. 5. Simplify $(\frac{1}{2} + \frac{1}{2}\sqrt{-3})^3$.

We are to use $\frac{1}{2}(1 + \sqrt{3}\sqrt{-1})$ three times as a factor. Reserving $(\frac{1}{2})^3 = \frac{1}{8}$ as the final coefficient, we have,

$$\begin{array}{l} 1 + \sqrt{3}\sqrt{-1} \\ \underline{1 + \sqrt{3}\sqrt{-1}} \\ 1 + \sqrt{3}\sqrt{-1} \\ \underline{\phantom{1 +{}} \sqrt{3}\sqrt{-1} - 3} \\ 1 + 2\sqrt{3}\sqrt{-1} - 3 \end{array} \qquad \begin{array}{l} -2 + 2\sqrt{3}\sqrt{-1} \\ \underline{1 + \sqrt{3}\sqrt{-1}} \\ -2 + 2\sqrt{3}\sqrt{-1} \\ \underline{\phantom{-2 +{}} -2\sqrt{3}\sqrt{-1} - 6} \\ -2 \phantom{+ 2\sqrt{3}\sqrt{-1}} - 6 = -8. \end{array}$$

Hence $(\frac{1}{2} + \frac{1}{2}\sqrt{-3})^3 = \frac{1}{8}(-8) = -1$.

EXERCISES

Perform the following indicated operations.

1. $\sqrt{-16}+\sqrt{-9}+\sqrt{-25}$.

2. $\sqrt{-x^4}-\sqrt{-x^2}$.

3. $3+5\sqrt{-1}-2\sqrt{-1}$.

4. $(2+3\sqrt{-1})(3+2\sqrt{-1})$.

5. $(2+3\sqrt{-1})(2-3\sqrt{-1})$.

6. $(4+5\sqrt{-3})(4-5\sqrt{-3})$.

7. $(2\sqrt{2}-3\sqrt{-3})(3\sqrt{3}+2\sqrt{-2})$.

8. $(\sqrt{-3}+\sqrt{-2})(\sqrt{-3}-\sqrt{-2})$.

9. $(3\sqrt{5}+2\sqrt{-7})(2\sqrt{5}-3\sqrt{-7})$.

10. $(-\frac{1}{2}-\frac{1}{2}\sqrt{-3})(-\frac{1}{2}-\frac{1}{2}\sqrt{-3})^2$.

Rationalize the denominators of

11. $\dfrac{2}{1-\sqrt{-1}}$.

12. $\dfrac{3}{\sqrt{3}+\sqrt{-1}}$.

13. $\dfrac{1-\sqrt{-1}}{1+\sqrt{-1}}$.

14. $\dfrac{\sqrt{2}+\sqrt{-3}}{\sqrt{2}-\sqrt{-3}}$.

15. $\dfrac{5}{2-3\sqrt{-5}}$.

16. $\dfrac{x+y\sqrt{-1}}{x\sqrt{-1}-y}$.

17. Solve $x^4-1=0$ by factoring. Find four roots and verify each.

18. Solve $x^3+1=0$ by factoring and the quadratic formula. Find three roots and verify each.

19. Solve $x^3-1=0$ as in the preceding and verify each root.

20. Solve $x^6-1=0$ by factoring and the quadratic formula.

SOLUTION OF EQUATIONS CONTAINING RADICALS

196. Many equations containing radicals are reducible to equivalent rational equations of the first or second degree.

The method of solving such equations is shown in the following examples.

Ex. 1. Solve $$1 + \sqrt{x} = \sqrt{3 + x}. \qquad (1)$$

Squaring and transposing, $$2\sqrt{x} = 2. \qquad (2)$$

Dividing by 2 and squaring, $$x = 1. \qquad (3)$$

Substituting in (1), $$1 + 1 = \sqrt{3 + 1} = 2.$$

Observe that only principal roots are used in this example.

If (1) is written $$1 + \sqrt{x} = -\sqrt{3 + x}, \qquad (4)$$

then (2) and (3) follow as before, but $x = 1$ does *not satisfy* (4). Indeed *algebra furnishes no means whereby to obtain a number which will satisfy* (4).

Ex. 2. Solve $$\sqrt{x + 5} = x - 1. \qquad (1)$$

Squaring and transposing, $$x^2 - 3x - 4 = 0. \qquad (2)$$

Solving, $x = 4$ and $x = -1$.

$x = 4$ satisfies (1) if the *principal* root in $\sqrt{x + 5}$ is taken. $x = -1$ does not satisfy (1) as it stands but would if the *negative* root were taken.

Ex. 3. Solve $$\frac{\sqrt{4x + 1} - \sqrt{3x - 2}}{\sqrt{4x + 1} + \sqrt{3x - 2}} = \frac{1}{5}. \qquad (1)$$

Clearing of fractions and combining similar radicals.

$$2\sqrt{4x + 1} = 3\sqrt{3x - 2}. \qquad (2)$$

Squaring and solving, we find $x = 2$.

This value of x satisfies (1) when *all* the roots are taken *positive* and also when all are taken *negative*, but otherwise *not*.

Ex. 4. Solve $\sqrt{2x+3} = \dfrac{3x-1}{\sqrt{3x}-1} - 1.$ (1)

The fraction in the second member should be reduced as follows:

$$\frac{3x-1}{\sqrt{3x}-1} = \frac{(\sqrt{3x}-1)(\sqrt{3x}+1)}{\sqrt{3x}-1} = \sqrt{3x}+1.$$

Hence, (1) reduces to $\sqrt{2x+3} = \sqrt{3x} + 1 - 1 = \sqrt{3x}.$ (2)

Solving, $x = 3$, which satisfies (1).

If we clear (1) of fractions in the ordinary manner, we have

$$(\sqrt{3x}-1)\sqrt{2x+3} = -\sqrt{3x} + 3x. \qquad (2')$$

Squaring both sides and transposing all rational terms to the second member,

$$2x\sqrt{3x} - 6\sqrt{3x} = 3x^2 - 8x - 3. \qquad (3)$$

Factoring each member,

$$2(x-3)\sqrt{3x} = (x-3)(3x+1), \qquad (4)$$

which is satisfied by $x = 3.$

Dividing each member by $x - 3$, squaring and transposing, we have

$$9x^2 - 6x + 1 = (3x-1)^2 = 0, \qquad (5)$$

which is satisfied by $x = \frac{1}{3}.$

Equation (1) is *not* satisfied by $x = \frac{1}{3}$, since the fraction in the second member is reduced to $\frac{0}{0}$ by this substitution. See § 50. The root $x = \frac{1}{3}$ is *introduced by clearing of fractions without first reducing the fraction to its lowest terms*, $\sqrt{3x}-1$ being a factor of both the numerator and the denominator. See § 165.

Ex. 5. Solve $\dfrac{6-x}{\sqrt{6-x}} - \sqrt{3} = \dfrac{x-3}{\sqrt{x-3}}.$ (1)

Reducing the fractions by removing common factors, we have

$$\sqrt{6-x} - \sqrt{3} = \sqrt{x-3}. \qquad (2)$$

Squaring, transposing, and squaring again,

$$x^2 - 9x + 18 = 0, \qquad (3)$$

whence $x = 3, \; x = 6.$

But neither of these is a root of (1). In this case (1) has *no* root.

197. In solving an equation containing radicals, we note the following:

(1) If a fraction of the form $\dfrac{a-b}{\sqrt{a}-\sqrt{b}}$ is involved, this should be reduced by dividing numerator and denominator by $\sqrt{a}-\sqrt{b}$ *before* clearing of fractions.

(2) After clearing of fractions, transpose terms so as to leave one radical alone in one member.

(3) Square both members, and if the resulting equation still contains radicals, transpose and square as before.

(4) In every case verify all results by substituting in the given equation. In case any value does not satisfy the given equation, determine whether the roots could be so taken that it would. See Ex. 3.

EXERCISES

Solve the following equations:

1. $\sqrt{x^2+7x-2}-\sqrt{x^2-3x+6}=2.$

2. $\sqrt{3y}-\sqrt{3y-7}=\dfrac{5}{\sqrt{3y-7}}.$

3. $\dfrac{by-1}{\sqrt{by}+1}=\dfrac{\sqrt{by}-1}{2}+4.$

4. $\sqrt{5x-19}+\sqrt{3x+4}=9.$

5. $\dfrac{\sqrt{x^2+a^2}-x}{\sqrt{x^2+a^2}+x}=2.$

6. $\sqrt{a+\sqrt{ax+x^2}}=\sqrt{a}-\sqrt{x}.$

7. $\dfrac{y-l}{\sqrt{y}+\sqrt{l}}=\dfrac{\sqrt{y}-\sqrt{l}}{3}+2\sqrt{l}.$

8. $\sqrt{5x}+1=1-\dfrac{5x-1}{\sqrt{5x}+1}.$

9. $\dfrac{4}{x}-\dfrac{\sqrt{4-x^2}}{x}=\sqrt{3}.$

10. $\dfrac{4+x+\sqrt{8x+x^2}}{4+x-\sqrt{8x+x^2}}=4.$

11. $\dfrac{a-x}{\sqrt{a-x}}+\dfrac{x+a}{\sqrt{x+a}}=\sqrt{a-b}.$

12. $\dfrac{x-a}{\sqrt{x}-\sqrt{a}}=\dfrac{\sqrt{x}+\sqrt{a}}{2}+2\sqrt{a}$

13. $\dfrac{m-y}{\sqrt{m-y}}-\sqrt{m-n}=-\dfrac{y-n}{\sqrt{y-n}}$.

14. $2\sqrt{x-a}+3\sqrt{2x}=\dfrac{7a+5x}{\sqrt{x-a}}$.

15. $\sqrt{2x+7}+\sqrt{2x+14}=\sqrt{4x+35+2\sqrt{4x^2+42x-21}}$.

16. $\sqrt{x-3}+\sqrt{x+9}=\sqrt{x+18}+\sqrt{x-6}$.

17. $\sqrt{x+7}-\sqrt{x-1}=\sqrt{x+2}+\sqrt{x-2}$.

18. $a\sqrt{y+b}-c\sqrt{b-y}=\sqrt{b(a^2+c^2)}$.

19. $y\sqrt{y-c}-\sqrt{y^3+c^3}+c\sqrt{y+c}=0$.

20. $\dfrac{\sqrt{x}}{\sqrt{m}}+\dfrac{\sqrt{m}}{\sqrt{x}}=\dfrac{\sqrt{m}}{\sqrt{n}}+\dfrac{\sqrt{n}}{\sqrt{m}}$.

21. $\sqrt{14+\sqrt{x}}+\sqrt{6-\sqrt{x}}=\dfrac{12}{\sqrt{6-\sqrt{x}}}$.

22. $\sqrt{3x}+\sqrt{3x+13}=\dfrac{91}{\sqrt{3x+13}}$.

23. $\sqrt{6x+3}+\sqrt{x+3}=2x+3$.

24. $\sqrt{x-a}+\sqrt{b-x}=\sqrt{b-a}$.

25. $\dfrac{\sqrt{x-a}+\sqrt{x-b}}{\sqrt{x-a}-\sqrt{x-b}}=\sqrt{\dfrac{x-a}{x-b}}$.

26. $\sqrt{(x-1)(x-2)}+\sqrt{(x-3)(x-4)}=\sqrt{2}$.

27. $\sqrt{2x+2}+\sqrt{7+6x}=\sqrt{7x+72}$.

28. $\sqrt{2abx}+\sqrt{a^2-bx}=\sqrt{a^2+bx}$.

29. $\dfrac{a+x+\sqrt{a^2-x^2}}{a+x-\sqrt{a^2-x^2}}=\dfrac{c}{x}$.

30. $\sqrt{x^2-2x+4}+\sqrt{3x^2+6x+12}=2\sqrt{x^2+x+10}$.

PROBLEMS

1. Find the altitude drawn to the longest side of the triangle whose sides are 6, 7, 8.

HINT. See figure, p. 235, E. C. Calling x and $8 - x$ the segments of the base and h the altitude, set up and solve two equations involving x and h.

2. Find the area of a triangle whose sides are 15, 17, 20.

First find one altitude as in problem 1.

3. Find the area of a triangle whose base is 16 and whose sides are 10 and 14.

4. Find the altitude on a side a of a triangle two of whose sides are a and a third b.

A three-sided pyramid all of whose edges are equal is called a regular tetrahedron. In Figure 10 AB, AC, AD, BC, BD, CD are all equal.

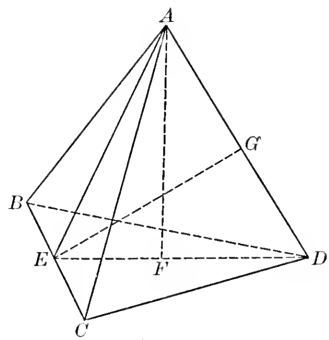

FIG. 10.

5. Find the altitude of a regular tetrahedron whose edges are each 6. Also the area of the base.

HINT. First find the altitudes AE and DE and then find the altitude of the triangle AED on the side DE, *i.e.* find AF.

6. Find the volume of a regular tetrahedron whose edges are each 10.

The volume of a tetrahedron is $\frac{1}{3}$ the product of the base and the altitude.

7. Find the volume of a regular tetrahedron whose edges are a.

8. In Figure 10 find EG if the edges are a.

9. If in Figure 10 EG is 12, compute the volume.

Use problem 8 to find the edge, then use problem 7 to find the volume.

10. Express the volume of the tetrahedron in terms of EG. That is if $EG=b$, find a general expression for the volume in terms of b.

11. If the altitude of a regular tetrahedron is 10, compute the edge accurately to two places of decimals.

12. Express the edge of a regular tetrahedron in terms of its altitude.

13. Express the volume of a regular tetrahedron in terms of its altitude.

14. Express the edge of a regular tetrahedron in terms of its volume.

15. Express the altitude of a regular tetrahedron in terms of its volume.

16. Express EG of Figure 10 in terms of the volume of the tetrahedron.

17. Find the edge of a regular tetrahedron such that its volume multiplied by $\sqrt{2}$, plus its entire surface multiplied by $\sqrt{3}$, is 144.

The resulting equation is of the third degree. Solve by factoring.

18. An electric light of 32 candle power is 25 feet from a lamp of 6 candle power. Where should a card be placed between them so as to receive the same amount of light from each?

Compare problem 13, p. 141. Compute result accurately to two places of decimals.

19. Where must the card be placed in problem 18 if the lamp is between the card and the electric light?

Notice that the roots of the equations in 18 and 19 are the same. Explain what this means.

20. State and solve a general problem of which 18 and 19 are special cases.

21. If the distance between the earth and the sun is 93 million miles, and if the mass of the sun is 300,000 times that of the earth, find two positions in which a particle would be equally attracted by the earth and the sun.

The gravitational attraction of one body upon another varies *inversely* as the square of the distance and directly as the product of the masses. Represent the mass of the earth by unity.

22. Find the volume of a pyramid whose altitude is 7 and whose base is a regular hexagon whose sides are 7.

The volume of a pyramid or a cone is $\frac{1}{3}$ the product of its base and its altitude.

23. If the volume of the pyramid in problem 22 were 100 cubic inches, what would be its altitude, a side of the base and the altitude being equal? Approximate the result to two places of decimals.

24. Express the altitude of the pyramid in problem 22 in terms of its volume, the altitude and the sides of the base being equal.

25. If in a right prism the altitude is equal to a side of the base, find the volume, the base being an equilateral triangle whose sides are a.

The volume of a right prism or cylinder equals the product of its base and its altitude.

26. Find the volume of the prism in problem 25 if its base is a regular hexagon whose side is a.

27. Express the side of the base of the prism in problem 25 in terms of its volume. State and solve a particular problem by means of the formula thus obtained.

28. Express the side of the base of the prism in problem 26 in terms of its volume. State and solve a particular problem by means of the formula thus obtained.

In Figures 11 and 12 the altitudes are each supposed to be three times the side a of the regular hexagonal bases.

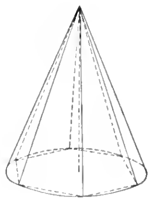

FIG. 11.

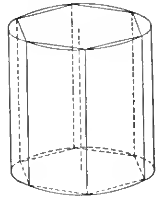

FIG. 12.

29. Express the difference between the volumes of the pyramid and the circumscribed cone in terms of a.

The volume of a cone equals $\frac{1}{3}$ the product of its base and altitude.

30. Express a in terms of the difference between the volumes of the cone and pyramid. State and solve a particular problem by means of the formula thus obtained.

31. Express the volume of the pyramid in terms of the difference between the areas of the bases of the cone and the pyramid. State a particular case and solve by means of the formula first obtained.

The lateral area of a right cylinder or prism equals the perimeter of the base multiplied by the altitude.

32. Express the difference of the lateral areas of the cylinder and the prism in terms of a.

The following four problems refer to Figure 12. In each case state a particular problem and solve by means of the formula obtained.

33. Express a in terms of the difference of the lateral areas.

34. Express the volume of the prism in terms of the difference of the perimeters of the bases.

35. Express the volume of the cylinder in terms of the difference of the lateral areas.

36. Express the sum of the volumes of the prism and cylinder in terms of the difference of the areas of the bases.

CHAPTER XI

LOGARITHMS

198. The operations of multiplication, division, and finding powers and roots are greatly shortened by the use of **logarithms.**

The logarithm of a number, in the system commonly used, is the *index of that power of* 10 *which equals the given number.*

Thus, 2 is the logarithm of 100 since $10^2 = 100$.

This is written $\log 100 = 2$.

Similarly $\log 1000 = 3$, since $10^3 = 1000$,

and $\log 10000 = 4$, since $10^4 = 10000$.

The logarithm of a number which is *not an exact* rational *power* of 10 is an irrational number and is written approximately as a decimal fraction.

Thus, $\log 74 = 1.8692$ since $10^{1.8692} = 74$ approximately.

In higher algebra it is shown that the laws for rational exponents (§ 179) hold also for irrational exponents.

199. The decimal part of a logarithm is called the **mantissa,** and the integral part the **characteristic.**

Since $10^0 = 1$, $10^1 = 10$, $10^2 = 100$, $10^3 = 1000$, etc., it follows that for all numbers between 1 and 10 the logarithm lies between 0 and 1, that is, the characteristic is 0. Likewise for numbers between 10 and 100 the characteristic is 1, for numbers between 100 and 1000 it is 2, etc.

200. Tables of logarithms (see p. 170) usually give the mantissas only, the characteristics being supplied, in the case of *whole* numbers, according to § 199, and in the case of decimal numbers, as shown in the examples given under § 201.

201. An important property of logarithms is illustrated by the following:

From the table of logarithms, p. 170, we have:

$$\log 376 = 2.5752, \text{ or } 376 = 10^{2.5752}. \qquad (1)$$

Dividing both members of (1) by 10 we have

$$37.6 = 10^{2.5752} \div 10^1 = 10^{2.5752-1} = 10^{1.5752}.$$

Hence, $\log 37.6 = 1.5752.$

Similarly, $\log 3.76 = 1.5752 - 1 = 0.5752.$

$$\log .376 = 0.5752 - 1, \text{ or } \bar{1}.5752,$$

$$\log .0376 = 0.5752 - 2, \text{ or } \bar{2}.5752,$$

where $\bar{1}$ and $\bar{2}$ are written for -1 and -2 to indicate that the characteristics are negative while the mantissas are positive.

Multiplying (1) by 10 gives

$$\log 3760 \ = 2.5752 + 1 = 3.5752,$$

and $\log 37600 = 2.5752 + 2 = 4.5752.$

Hence, if the decimal point of a number is moved a certain number of places to the *right* or to the *left*, the characteristic of the logarithm is *increased* or *decreased* by a corresponding number of units, the mantissa *remaining the same*.

From the table on pp. 170, 171, we may find the mantissas of logarithms for all integral numbers from 1 to 1000. In this table the logarithms are given to four places of decimals, which is sufficiently accurate for most practical purposes.

E.g. for log 4 the mantissa is the same as that for log 40 or for log 400.

To find log .0376 we find the mantissa corresponding to 376, and prefix the characteristic $\bar{2}$. See above.

Ex. 1. Find log 876.

Solution. Look down the column headed N to 87, then along this line to the column headed 6, where we find the number 9425, which is the mantissa. Hence $\log 876 = 2.9425$.

Ex. 2. Find log 3747.

Solution. As above we find log 3740 = 3.5729,

and log 3750 = 3.5740.

The difference between these logarithms is 11, which corresponds to a difference of 10 between the numbers. But 3740 and 3747 differ by 7. Hence, their logarithms should differ by $\frac{7}{10}$ of 11, *i.e.* by 8.1. Adding this to the logarithm of 3740, we have 3.5737, which is the required logarithm.

The assumption here made, that the logarithm varies directly as the number, is not quite, but very nearly, accurate, when the variation of the number is confined to a narrow range, as is here the case.

Ex. 3. Find the number whose logarithm is 2.3948.

Solution. Looking in the table, we find that the nearest *lower* logarithm is 2.3945 which corresponds to the number 248. See § 199.

The given mantissa is 3 greater than that of 248, while the mantissa of 249 is 17 greater. Hence the number corresponding to 2.3948 must be 248 plus $\frac{3}{17}$ or .176. Hence, 248.18 is the required number, correct to 2 places of decimals.

Ex. 4. Find log .043.

Solution. Find log 43 and subtract 3 from the characteristic.

Ex. 5. Find the number whose logarithm is $\bar{4}.3949$.

Solution. Find the number whose logarithm is 0.3949, and move the decimal point 4 places to the left.

EXERCISES

Find the logarithms of the following numbers:

1. 491.	6. .541.	11. .006.	16. 79.31.
2. 73.5.	7. .051.	12. .1902.	17. 4.245.
3. 2485.	8. 8104.	13. .0104.	18. .0006.
4. 539.7.	9. 70349.	14. 2.176.	19. 3.817.
5. 53.27.	10. 439.26.	15. 8.094.	20. .1341.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	7857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

Find the numbers corresponding to the following logarithms:

21. 1.3179.	26. 2.9900.	31. $\bar{1}$.5972.	36. 0.2468.
22. 3.0146.	27. 0.1731.	32. 1.0011.	37. 0.1357.
23. 0.7145.	28. 0.8974.	33. $\bar{2}$.7947.	38. $\bar{2}$.0246.
24. $\bar{1}$.5983.	29. 0.9171.	34. $\bar{2}$.5432.	39. 1.1358.
25. 2.0013.	30. 3.4015.	35. 0.5987.	40. 4.0478.

202. **Products and powers** may be found by means of logarithms, as shown by the following examples.

Ex. 1. Find the product $49 \times 134 \times .071 \times 349$.

Solution. From the table,

$$\log 49 = 1.6902 \text{ or } 49 = 10^{1.6902}.$$
$$\log 134 = 2.1271 \text{ or } 134 = 10^{2.1271}.$$
$$\log .071 = \bar{2}.8513 \text{ or } .071 = 10^{\bar{2}.8513}.$$
$$\log 349 = 2.5428 \text{ or } 349 = 10^{2.5428}.$$

Since powers of the same base are multiplied by *adding* exponents, § 176, we have $49 \times 134 \times .071 \times 349 = 10^{5.2114}$.

Hence $\log(49 \times 134 \times .071 \times 349) = 5.2114$.

The number corresponding to this logarithm, as found by the method used in Ex. 3 above, is 162701. By actual multiplication the product is found to be 162698.911 or 162699 which is the nearest approximation without decimals. Hence the product obtained by means of logarithms is 5 too large. This is an error of about $\frac{1}{32000}$ of the actual result and is therefore so small as to be negligible.

Ex. 2. Find $(1.05)^{20}$.

Solution. $\text{Log } 1.05 = 0.0212$ or $10^{0.0212} = 1.05$.

Hence $(1.05)^{20} = (10^{0.0212})^{20} = 10^{0.0212 \cdot 20} = 10^{0.424}$,

or $\log(1.05)^{20} = 0.4240$.

Hence $(1.05)^{20}$ is the number corresponding to the logarithm 0.4240.

Since logarithms are **exponents** of the base 10, it follows from the laws of exponents (see § 198) that

(*a*) *The logarithm of the product of two or more numbers is the sum of the logarithms of the numbers.*

(*b*) *The logarithm of a power of a number is the logarithm of the number multiplied by the index of the power.*

That is,

$$\log(a \cdot b \cdot c) = \log a + \log b + \log c, \text{ and } \log a^n = n \log a.$$

EXERCISES

By means of the logarithms obtain the following products and powers:

1. $243 \times 76 \times .34$.	7. 5.93×10.02.	13. $(49)^3 \times .19 \times 21^2$.
2. 823.68×370.	8. 486×3.45.	14. $.21084 \times (.53)^2$.
3. 216.83×2.03.	9. $(.02)^2 \times (0.8)$.	15. $7.865 \times (.013)^2$.
4. $57^2 \times (.71)^2$.	10. $(65)^2 \times (91)^3$.	16. $(6.75)^3 \times (723)^2$.
5. $510 \times (9.1)^3$.	11. $(84)^2 \times (75)^3$.	17. $(1.46)^2 \times (61.2)^2$.
6. $43.71 \times (21)^2$.	12. $(.960)^2(49)^2$.	18. $(3.54)^3 \times (29.3)^2$.

19. $(4.132)^2 \times (5.184)^2$. 20. $1946 \times 398 \times .08$.

203. Quotients and roots may be found by means of logarithms, as shown by the following examples.

Ex. 1. Divide 379 by 793.

Solution. From the table,

$$\log 379 = 2.5786 \text{ or } 10^{2.5786} = 379.$$

$$\log 793 = 2.8993 \text{ or } 10^{2.8993} = 793.$$

Hence by the law of exponents for division, § 175,

$$379 \div 793 = 10^{2.5786 - 2.8993}.$$

Since in all operations with logarithms the mantissa is positive, write the first exponent $3.5786 - 1$ and then subtract 2.8993.

Hence $\log(379 \div 793) = .6793 - 1 = \bar{1}.6793$.

Hence the quotient is the number corresponding to this logarithm.

Ex. 2. By means of logarithms approximate $\sqrt[3]{42^2 \times 37^5}$.

By the methods used above we find

$$\log(42^2 \times 37^5) = 11.0874 \text{ or } 10^{11.0874} = 42^2 \times 37^5.$$

Hence $$\sqrt[3]{42^2 \times 37^5} = (10^{11.0874})^{\frac{1}{3}} = 10^{\frac{11.0874}{3}} = 10^{3.6958}.$$

That is, $$\log \sqrt[3]{42^2 \times 37^5} = 3.6958.$$

Hence the result sought is the number corresponding to this logarithm.

It follows from the laws of exponents (see § 198) that

(*a*) *The logarithm of a quotient equals the logarithm of the dividend minus the logarithm of the divisor.*

(*b*) *The logarithm of a root of a number is the logarithm of the number divided by the index of the root.*

That is

$$\log \frac{a}{b} = \log a - \log b \text{ and } \log \sqrt[n]{a} = \frac{\log a}{n}.$$

EXERCISES

By means of logarithms approximate the following quotients and roots:

1. $45.2 \div 8.9$.
2. $231.18 \div 4.2$.
3. $.04905 \div .327$.
4. $\sqrt{196 \times 256}$.
5. $\dfrac{5334 \times .02374}{27.43 \times 3.246}$.
6. $\sqrt[5]{69} \div \sqrt[3]{21}$.
7. $\sqrt[7]{15} \times \sqrt[8]{67}$.
8. $\sqrt[10]{211} \times \sqrt[11]{34.7}$.
9. $(5184)^{\frac{1}{2}} \div (38124)^{\frac{1}{3}}$.
10. $(6.75)^3 \div (2.132)^2$.
11. $\sqrt[9]{105} \div \sqrt[12]{76}$.
12. $(91125)^{\frac{1}{3}} \div (576)^{\frac{1}{2}}$.
13. $(3.040)^3 \div (.0065)^3$.
14. $(29.3)^{\frac{1}{3}} \div \sqrt{(3.47)^3}$.
15. $\sqrt[3]{39} \times \sqrt[3]{56} \times \sqrt[4]{87}$.
16. $\sqrt[3]{\dfrac{13^4 \times .31^2 \times 4.31^3}{\sqrt{71} \times \sqrt[3]{41} \times \sqrt{51}}}$.
17. $\sqrt[5]{\dfrac{4^9 \times .57^3 \times 42^3}{\sqrt[3]{5.2} \times \sqrt{.83} \times \sqrt{23}}}$.
18. $\left(\dfrac{\sqrt[3]{54} \times \sqrt[4]{28} \times \sqrt[5]{7}}{\sqrt[2]{4^7} \times \sqrt[3]{7^4} \times (.003)^{\frac{1}{4}}}\right)^{\frac{4}{5}}$.

CHAPTER XII

PROGRESSIONS

ARITHMETIC PROGRESSIONS

204. An arithmetic progression is a series of numbers, such that any one after the first is obtained by adding a fixed number to the preceding. The fixed number is called the **common difference**.

The general form of an arithmetic progression is

$$a, \quad a+d, \quad a+2\,d, \quad a+3\,d, \quad \cdots,$$

where a is the first term and d the common difference.

E.g. 2, 5, 8, 11, 14, ⋯ is an arithmetic progression in which 2 is the first term and 3 the common difference. Written in the general form, it would be $2, \ 2+3, \ 2+2\cdot 3, \ 2+3\cdot 3, \ 2+4\cdot 3, \ \cdots.$

205. If there are n terms in the progression, then the last term is $a+(n-1)d$. Indicating the last term by l, we have

$$l = a+(n-1)d. \qquad \text{I}$$

An arithmetic progression of n terms would then be written in general form, thus,

$$a, \quad a+d, \quad a+2\,d, \quad \cdots, \quad a+(n-2)d, \quad a+(n-1)d.$$

EXERCISES

1. Solve I for each letter in terms of all the others.

In each of the following find the value of the letter not given, and write out the progression in each case.

2. $\begin{cases} a=2, \\ d=2, \\ n=7. \end{cases}$ 3. $\begin{cases} a=3, \\ d=5, \\ l=43. \end{cases}$ 4. $\begin{cases} a=1, \\ n=15, \\ l=15. \end{cases}$ 5. $\begin{cases} a=7, \\ n=31, \\ l=91. \end{cases}$

6. $\begin{cases} a = 4, \\ d = -3, \\ n = 18. \end{cases}$ 8. $\begin{cases} a = 3, \\ d = -5, \\ l = -32. \end{cases}$ 10. $\begin{cases} d = -5, \\ n = 13, \\ l = -63. \end{cases}$ 12. $\begin{cases} a = 11, \\ l = -39, \\ d = -5. \end{cases}$

7. $\begin{cases} a = -5, \\ d = 4, \\ n = 7. \end{cases}$ 9. $\begin{cases} d = 7, \\ n = 8, \\ l = 24. \end{cases}$ 11. $\begin{cases} a = -3, \\ n = 9, \\ l = -27. \end{cases}$ 13. $\begin{cases} a = x, \\ l = y, \\ n = z. \end{cases}$

206. The sum of an arithmetic progression of n terms may be obtained as follows:

Let s_n denote the sum of n terms of the progression. Then,

$$s_n = a + [a + d] + [a + 2d] + \cdots + [a + (n-2)d] + [a + (n-1)d]. \quad (1)$$

This may also be written, reversing the order of the terms, thus,

$$s_n = [a + (n-1)d] + [a + (n-2)d] + \cdots + [a + 2d] + [a + d] + a. \quad (2)$$

Adding (1) and (2), we have

$$2s_n = [2a + (n-1)d] + [2a + (n-2)d + d] + \cdots + [2a + (n-2)d + d] + [2a + (n-1)d].$$

The expression in each bracket is reducible to $2a + (n-1)d$, which may also be written $a + [a + (n-1)d] = a + l$, by § 205.

Since there are n of these expressions, each $a + l$, we have

$$2s_n = n(a + l).$$

Hence $$s_n = \frac{n}{2}(a + l). \qquad \text{II}$$

This formula for the sum of n terms involves a, l, and n, that is, the first term, the last term, and the number of terms.

207. In the two equations,

$$l = a + (n-1)d, \qquad \text{I}$$

$$s = \frac{n}{2}(a + l), \qquad \text{II}$$

there are five letters, namely, a, d, l, n, s. If any three of these are given, the equations I and II may be solved simultaneously to find the other two, considered as the *unknowns*.

The solution of problems in arithmetic progression by means of equations I and II is illustrated in the following examples:

Ex. 1. Given $n = 11$, $l = 23$, $s = 143$. Find a and d.

Substituting the given values in I and II,

$$23 = a + (11 - 1)d. \qquad (1)$$

$$143 = \tfrac{11}{2}(a + 23). \qquad (2)$$

From (2), $a = 3$, which in (1) gives $d = 2$.

Ex. 2. Given $d = 4$, $n = 5$, $s = 75$. Find a and l.

From I and II, $$l = a + (5 - 1)4, \qquad (1)$$

$$75 = \frac{5}{2}(a + l). \qquad (2)$$

Solving (1) and (2) simultaneously, we have $a = 7$, $l = 23$.

Ex. 3. Given $d = 4$, $l = 35$, $s = 161$. Find a and n.

From I and II, $$35 = a + (n - 1)4, \qquad (1)$$

$$161 = \frac{n}{2}(a + 35). \qquad (2)$$

From (1) $$a = 39 - 4n,$$

which in (2) gives $$161 = \frac{n}{2}(74 - 4n) = 37n - 2n^2.$$

Hence $$n = \tfrac{23}{2}, \text{ or } 7.$$

Since an arithmetic progression must have an *integral* number of terms, only the second value is applicable to this problem.

Ex. 4. Given $d = 2$, $l = 11$, $s = 35$. Find a and n.

Substituting in I and II, and solving for a and n, we have

$$a = 3,\ n = 5, \text{ and } a = -1,\ n = 7.$$

Hence there are two progressions,

$$-1, 1, 3, 5, 7, 9, 11,$$

and $$3, 5, 7, 9, 11,$$

each of which satisfies the given conditions.

EXERCISES

In each of the following obtain the values of the two letters not given.

If fractional or negative values of n are obtained, such a result indicates that the problem is impossible. This is also the case if an *imaginary* value is obtained for *any* letter. In each exercise interpret all the values found.

1. $\begin{cases} s=96, \\ l=19, \\ d=2. \end{cases}$
2. $\begin{cases} s=34, \\ l=14, \\ d=3. \end{cases}$
3. $\begin{cases} a=7, \\ l=27, \\ s=187. \end{cases}$
4. $\begin{cases} s=88, \\ l=-7, \\ d=-3. \end{cases}$
5. $\begin{cases} n=18, \\ a=4, \\ l=13. \end{cases}$
6. $\begin{cases} n=14, \\ a=7, \\ s=14. \end{cases}$
7. $\begin{cases} d=-1, \\ n=11, \\ l=-35. \end{cases}$
8. $\begin{cases} l=30, \\ s=162, \\ n=9. \end{cases}$
9. $\begin{cases} a=30, \\ n=10, \\ s=120. \end{cases}$
10. $\begin{cases} d=6, \\ l=49, \\ s=232. \end{cases}$
11. $\begin{cases} s=7, \\ d=1\frac{1}{2}, \\ l=7. \end{cases}$
12. $\begin{cases} s=14, \\ d=3, \\ l=4. \end{cases}$

In each of the following call the two letters specified the *unknowns* and solve for their values in terms of the remaining three letters supposed to be *known*.

13. $a, d.$ 15. $a, n.$ 17. $d, l.$ 19. $d, s.$ 21. $l, s.$

14. $a, l.$ 16. $a, s.$ 18. $d, n.$ 20. $l, n.$ 22. $n, s.$

208. **Arithmetic means.** The terms between the first and the last of an arithmetic progression are called **arithmetic means.**

Thus, in 2, 5, 8, 11, 14, 17, the four arithmetic means between 2 and 17 are 5, 8, 11, 14.

If the first and the last terms and the number of arithmetic means between them are given, then these means can be found.

For we have given a, l, and n. Hence d can be found and the whole series constructed.

Example. Insert 7 arithmetic means between 3 and 19.

In this progression $a = 3$, $l = 19$, and $n = 9$.

Hence from $l = a + (n - 1)d$ we find $d = 2$ and the required means are 5, 7, 9, 11, 13, 15, 17.

209. The case of *one* arithmetic mean is important. Let A be the arithmetic mean between a and l. Since a, A, l are in arithmetic progression, we have $A = a + d$, and $l = A + d$. Hence

$$A - l = a - A$$

or

$$A = \frac{a + l}{2}. \qquad \text{III}$$

EXERCISES AND PROBLEMS

1. Insert 5 arithmetic means between 5 and -7.
2. Insert 3 arithmetic means between -2 and 12.
3. Insert 8 arithmetic means between -3 and -5.
4. Insert 5 arithmetic means between -11 and 40.
5. Insert 15 arithmetic means between 1 and 2.
6. Insert 9 arithmetic means between $2\frac{3}{4}$ and $-1\frac{1}{2}$.
7. Find the arithmetic mean between 3 and 17.
8. Find the arithmetic mean between -4 and 16.
9. Find the tenth and eighteenth terms of the series 4, 7, 10, $\cdots$.
10. Find the fifteenth and twentieth terms of the series -8, -4, 0, $\cdots$.
11. The fifth term of an arithmetic progression is 13 and the thirtieth term is 49. Find the common difference.
12. Find the sum of all the integers from 1 to 100.
13. Find the sum of all the odd integers between 0 and 200.
14. Find the sum of all integers divisible by 6 between 1 and 500.
15. Show that $1 + 3 + 5 + \cdots + n = n^2$ where n is any odd integer.

16. In a potato race 40 potatoes are placed in a straight line one yard apart, the first potato being two yards from the basket. How far must a contestant travel in bringing them to the basket one at a time?

17. There are three numbers in arithmetic progression whose sum is 15. The product of the first and last is $3\frac{1}{5}$ times the second. Find the numbers.

18. There are four numbers in arithmetic progression whose sum is 20 and the sum of whose squares is 120. Find the numbers.

19. If a body falls from rest 16.08 feet the first second, 48.24 feet the second second, 80.40 the third, etc., how far will it fall in 10 seconds? 15 seconds? t seconds?

20. According to the law indicated in problem 19 in how many seconds will a body fall 1000 feet? s feet?

If a body is thrown downward with a velocity of v_0 feet per second, then the distance, s, it will fall in t seconds is v_0t feet plus the distance it would fall if starting from rest.

That is, $\boldsymbol{s = v_0t + \frac{1}{2}gt^2}$, where $g = 32.16$.

21. In what time will a body fall 1000 feet if thrown downward with a velocity of 20 feet per second?

22. With what velocity must a body be thrown downward in order that it shall fall 360 feet in 3 seconds?

23. A stone is dropped into a well, and the sound of its striking the bottom is heard in 3 seconds. How deep is the well if sound travels 1080 feet per second?

A body thrown upward with a certain velocity will rise as far as it would have to fall to acquire this velocity. The velocity (neglecting the resistance of the atmosphere) of a body starting from rest is $\boldsymbol{gt}$ where $g = 32.16$ and t is the number of seconds.

24. A rifle bullet is shot directly upward with a velocity of 2000 feet per second. How high will it rise, and how long before it will reach the ground?

25. From a balloon 5800 feet above the earth, a body is thrown downward with a velocity of 40 feet per second. In how many seconds will it reach the ground?

26. If in Problem 25 the body is thrown upward at the rate of 40 feet per second, how long before it will reach the ground?

GEOMETRIC PROGRESSIONS

210. A **geometric progression** is a series of numbers in which any term after the first is obtained by multiplying the preceding term by a fixed number, called the **common ratio.**

The general form of a geometric progression is

$$a,\ ar,\ ar^2,\ ar^3,\ \cdots,\ ar^{n-1},$$

in which a is the first term, r the constant multiplier, or common ratio, and n the number of terms.

E.g. 3, 6, 12, 24, 48, is a geometric progression in which 3 is the first term, 2 is the common ratio, and 5 is the number of terms.

Written in the general form it would be $3,\ 3 \cdot 2,\ 3 \cdot 2^2,\ 3 \cdot 2^3,\ 3 \cdot 2^4$.

211. If l is the last or nth term of the series, then

$$l = ar^{n-1}. \qquad \text{I}$$

If any three of the four letters in I are given, the remaining one may be found by solving this equation.

EXERCISES

In each of the following find the value of the letter not given:

1. $\begin{cases} l = 162, \\ r = 3, \\ n = 5. \end{cases}$

2. $\begin{cases} a = 1, \\ r = 2, \\ n = 8. \end{cases}$

3. $\begin{cases} a = -4, \\ r = -3, \\ n = 6. \end{cases}$

4. $\begin{cases} a = -1, \\ r = -2, \\ n = 9. \end{cases}$

5. $\begin{cases} l = 1024, \\ r = -2, \\ n = 11. \end{cases}$

6. $\begin{cases} l = 1024, \\ r = 2, \\ n = 11. \end{cases}$

7. $\begin{cases} a = -\frac{1}{2}, \\ r = \frac{3}{2}, \\ n = 6. \end{cases}$

8. $\begin{cases} l = 18, \\ r = \frac{1}{3}, \\ n = 6. \end{cases}$

9. $\begin{cases} l = -16, \\ r = -\frac{3}{4}, \\ n = 5. \end{cases}$

10. $\begin{cases} l = 32, \\ r = -2, \\ n = 6. \end{cases}$

11. $\begin{cases} a = -2, \\ r = -\frac{3}{2}, \\ n = 7. \end{cases}$

12. $\begin{cases} a = 3, \\ r = 2, \\ l = 1536. \end{cases}$

212. The **sum of *n* terms** of a geometric expression may be found as follows:

If s_n denotes the sum of n terms, then

$$s_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}. \qquad (1)$$

Multiplying both members of (1) by r, we have

$$rs_n = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n. \qquad (2)$$

Subtracting (1) from (2), and canceling terms, we have

$$rs_n - s_n = ar^n - a. \qquad (3)$$

Solving (3) for s_n we have

$$\mathbf{s_n = \frac{ar^n - a}{r - 1} = \frac{a(r^n - 1)}{r - 1}}. \qquad \text{II}$$

This formula for the sum of n terms of a geometric series involves only a, r, and n.

Since $ar^n = r \cdot ar^{n-1} = r \cdot l$, s^n may also be written:

$$\mathbf{s_n = \frac{rl - a}{r - 1} = \frac{a - rl}{1 - r}}. \qquad \text{III}$$

This formula involves only r, l, and a.

213. From equations I and II or I and III any two of the numbers a, l, r, s, and n can be found when the other three are given, as in the following examples.

Ex. 1. Given $n = 7$, $r = 2$, $s = 381$. Find a and l.

From I and III, $\qquad l = a \cdot 2^6 = 64\,a, \qquad (1)$

$$381 = \frac{2\,l - a}{2 - 1} = 2\,l - a. \qquad (2)$$

Substituting $l = 64\,a$ in (2), we obtain $a = 3$, and $l = 192$.

Ex. 2. Given $a = -3$, $l = -243$, $s = -183$. Find r and n.

From I and III, $\qquad -243 = (-3)r^{n-1}, \qquad (1)$

$$-183 = \frac{-243\,r + 3}{r - 1}. \qquad (2)$$

From (2) $\qquad r = -3. \qquad (3)$

From (1) $\qquad 81 = (-3)^{n-1}. \qquad (4)$

Since $(-3)^4 = 81$, we have $n - 1 = 4$ or $n = 5$.

EXERCISES

1. Solve II for a in terms of the remaining letters.

2. Solve III for each letter in terms of the remaining letters.

In each of the following find the terms represented by the interrogation point.

3. $\begin{cases} a = 1, \\ r = 3, \\ n = 5, \\ s = ? \end{cases}$ 4. $\begin{cases} s = 635, \\ r = 2, \\ n = 7, \\ a = ? \end{cases}$ 5. $\begin{cases} s = 13, \\ r = \frac{2}{3}, \\ n = 4, \\ a = ? \end{cases}$ 6. $\begin{cases} l = -\frac{16}{81}, \\ s = ? \\ n = 5, \\ r = \frac{2}{3}. \end{cases}$

7. $\begin{cases} a = 1, \\ s = \frac{25}{64}, \\ l = -\frac{27}{64}, \\ r = ? \end{cases}$ 8. $\begin{cases} r = \frac{1}{6}, \\ n = 5, \\ l = 1296, \\ a = ? \\ s = ? \end{cases}$ 9. $\begin{cases} r = \frac{3}{2}, \\ n = 8, \\ s = 1050\frac{5}{6}, \\ l = ? \\ a = ? \end{cases}$ 10. $\begin{cases} a = \frac{9}{2}, \\ n = 7, \\ l = \frac{32}{81}, \\ r = ? \\ s = ? \end{cases}$

214. Geometric means. The terms between the first and the last of a geometric progression are called **geometric means.**

Thus in 3, 6, 12, 24, 48, three geometric means between 3 and 48 are 6, 12 and 24.

If the first term, the last term, and the number of geometric means are given, the ratio may be found from I, and then the means may be inserted.

Example. Insert 4 geometric means between 2 and 64.

We have given $a = 2$, $l = 64$, $n = 4 + 2 = 6$, to find r.

From I, $\qquad 64 = 2 \cdot r^{6-1}$ or $r^5 = 32$ and $r = 2$.

Hence, the series is 2, 4, 8, 16, 32, 64.

215. The case of *one* geometric mean is important. If G is the geometric mean between a and b, we have $\dfrac{G}{a} = \dfrac{b}{G}$.

Hence, $\qquad \boldsymbol{G = \sqrt{ab}}.$

216. Problem. In attempting to reduce $\frac{2}{3}$ to a decimal, we find by division $.666 \cdots$, the dots indicating that the process goes on indefinitely.

Conversely, we see that $.666 \cdots = \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} + \cdots$, that is, a geometric progression in which $a = \frac{6}{10}$, $r = \frac{1}{10}$, and n is not fixed but goes on *increasing indefinitely*.

As n grows large, l grows small, and by taking n sufficiently large, l can be made as small as we please. Hence formula III, § 212, is to be interpreted in this case as follows:

$$s_n = \frac{a - rl}{1 - r} = \frac{\frac{6}{10} - \frac{l}{10}}{1 - \frac{1}{10}} = \frac{6 - l}{9},$$

in which l grows small indefinitely as n increases indefinitely, so that by taking n large enough s_n can be made to differ as little as we please from $\frac{6 - 0}{9} = \frac{6}{9} = \frac{2}{3}$.

In this case we say s_n **approaches** $\frac{2}{3}$ **as a limit** as n increases indefinitely.

Observe that this interpretation can apply only when the constant multiplier r is a proper fraction.

EXERCISES AND PROBLEMS

1. Insert 5 geometric means between 2 and 128.
2. Insert 7 geometric means between 1 and $\frac{1}{256}$.
3. Find the geometric mean between 8 and 18.
4. Find the geometric mean between $\frac{1}{12}$ and $\frac{1}{4}$.
5. Find the fraction which is the limit of $.333 \cdots$.
6. Find the fraction which is the limit of $.1666 \cdots$.
7. Find the fraction which is the limit of $.08333 \cdots$.
8. Find the 13th term of $-\frac{16}{3}, 4, -3 \cdots$.
9. Find the sum of 15 terms of the series $-243, 81, -27 \cdots$.
10. Find the limit of the sum $\frac{4}{3} + \frac{2}{3} + \frac{1}{3} + \frac{1}{6} + \cdots$, as the number of terms increases indefinitely.

Given	Find	Given	Find
11. a, r, n	l, s	**15.** a, n, l	s, r
12. a, r, s	l	**16.** r, n, l	s, a
13. r, n, s	l, a	**17.** r, l, s	a
14. a, r, l	s	**18.** a, l, s	r

19. The product of three terms of a geometric progression is 1000. Find the second term.

20. Four numbers are in geometric progression. The sum of the second and third is 18, and the sum of the first and fourth is 27. Find the numbers.

21. Find an arithmetic progression whose first term is 1 and whose first, second, fifth, and fourteenth terms are in geometric progression.

22. Three numbers whose sum is 27 are in arithmetic progression. If 1 is added to the first, 3 to the second, and 11 to the third the sums will be in geometric progression. Find the numbers.

23. To find the compound interest when the principal, the rate of interest, and the time are given.

Solution. Let p equal the number of dollars invested, r the rate of per cent of interest, t the number of years, and a the amount at the end of t years.

Then $\quad a = p(1 + r)$ at the end of one year.

$\qquad a = p(1 + r)(1 + r) = p(1 + r)^2$ at the end of two years.

and $\qquad a = p(1 + r)^t$ at the end of t years.

That is, the amount for t years is the last term of a geometric progression in which p is the first term, $1 + r$ is the ratio, and $t + 1$ is the number of terms.

24. Show how to modify the solution given under problem 23 when the interest is compounded semiannually; quarterly.

25. Solve the equation $a = p(1 + r)^t$ for p and for r.

26. Solve $a = p(1 + r)^t$ for t.

Solution. $\log a = \log p(1 + r)^t = \log p + \log (1 + r)^t$

$= \log p + t \log (1 + r)$. (See § 202.) Hence $t = \dfrac{\log a - \log p}{\log (1 + r)}$.

27. At what rate of interest compounded annually will \$1200 amount to \$1800 in 12 years?

28. At what rate of interest compounded semiannually will a sum double itself in 20 years? in 15 years? in 10 years?

29. In what time will \$8000 amount to \$13,500, the rate of interest being $3\frac{1}{2}\%$ compounded annually?

30. In what time will a sum double itself at 3%, 4%, 5%, compounded semiannually?

The present value of a debt due at some future time is a sum such that, if invested at compound interest, the amount at the end of the time will equal the debt.

31. What is the present value of \$2500 due in 4 years, money being worth $3\frac{1}{2}\%$ interest compounded semiannually?

32. A man bequeathed \$50,000 to his daughter, payable on her twenty-fifth birthday, with the provision that the present worth of the bequest should be paid in case she married before that time. If she married at 21, how much would she receive, interest being 4% per annum and compounded quarterly?

33. What is the rate of interest if the present worth of \$21,000 due in 7 years is \$19,500?

34. In how many years is \$5000 due if its present worth is \$3500, the rate of interest being $3\frac{3}{4}\%$ compounded annually?

HARMONIC PROGRESSIONS

217. A **harmonic progression** is a series whose terms are the reciprocals of the corresponding terms of an arithmetic progression.

E.g. $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9} \cdots$ is a harmonic progression whose terms are the reciprocals of the terms of the arithmetic progression $1, 3, 5, 7, 9 \cdots$.

The name *harmonic* is given to such a series because musical strings of uniform size and tension, whose lengths are the reciprocals of the positive integers, *i.e.* $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$, vibrate in harmony.

The general form of the harmonic progression is

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \cdots, \frac{1}{a+(n-1)d}. \qquad \text{I}$$

It follows that if $a, b, c, d, e, \ldots$ are in harmonic progression, then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}, \frac{1}{e}, \cdots$ are in arithmetic progression. Hence, all questions pertaining to a harmonic progression are best answered by first converting it into an arithmetic progression.

218. Harmonic means. The terms between the first and the last of a harmonic progression are called **harmonic means** between them.

Example. Insert five harmonic means between 30 and 3.

This is done by inserting five arithmetic means between $\frac{1}{30}$ and $\frac{1}{3}$. By the method of § 208 the arithmetic series is found to be $\frac{1}{30}, \frac{1}{12}, \frac{2}{15}, \frac{11}{60}, \frac{7}{30}, \frac{17}{60}, \frac{1}{3}$. Hence, the harmonic series is $30, 12, \frac{15}{2}, \frac{60}{11}, \frac{30}{7}, \frac{60}{17}, 3$.

219. The case of a single harmonic mean is important. Let a, H, l be in harmonic progression. Then $\frac{1}{a}, \frac{1}{H}, \frac{1}{l}$ are in arithmetic progression.

Hence, by § 209, $\dfrac{1}{H} = \dfrac{\frac{1}{a}+\frac{1}{l}}{2}$ or $H = \dfrac{2al}{a+l}$.

220. The arithmetic, geometric, and harmonic means between a and l are related as follows:

We have seen $\quad A = \dfrac{a+l}{2},\ G = \sqrt{al},\ H = \dfrac{2al}{a+l}.$

Hence, $\quad \dfrac{A}{G^2} = \dfrac{a+l}{2} \div al = \dfrac{a+l}{2al}.$

Therefore, $\quad \dfrac{A}{G^2} = \dfrac{1}{H}$, or $\dfrac{A}{G} = \dfrac{G}{H}.$

That is, G is a *mean proportional* between A and H. See § 172.

EXERCISES AND PROBLEMS

1. Insert three harmonic means between 22 and 11.

2. Insert six harmonic means between $\frac{1}{3}$ and $\frac{23}{6}$.

3. The first term of a harmonic progression is $\frac{1}{2}$ and the tenth term is $\frac{1}{20}$. Find the intervening terms.

4. Two consecutive terms of a harmonic progression are 5 and 6. Find the next two terms and also the two preceding terms.

5. If a, b, c are in harmonic progression, show that $a \div c = (a - b) \div (b - c)$.

6. Find the arithmetic, geometric, and harmonic means between:

(*a*) 16 and 36; (*b*) $m + n$ and $m - n$; (*c*) $\dfrac{1}{m+n}$ and $\dfrac{1}{m-n}$.

7. The harmonic mean between two numbers exceeds their arithmetic mean by 7, and one number is three times the other. Find the numbers.

8. If x, y, and z are in arithmetic progression, show that mx, my, and mz are also in arithmetic progression.

9. x, y, and z being in harmonic progression, show that $\dfrac{x}{x+y+z}$, $\dfrac{y}{x+y+z}$, and $\dfrac{z}{x+y+z}$ are in harmonic progression, and also that $\dfrac{x}{y+z}$, $\dfrac{y}{x+z}$, and $\dfrac{z}{x+y}$ are in harmonic progression.

10. The sum of three numbers in harmonic progression is 3, and the first is double the third. Find the numbers.

11. The geometric mean between two numbers is $\frac{1}{4}$ and the harmonic mean is $\frac{1}{5}$. Find the numbers.

12. Insert n harmonic means between the numbers a and b.

CHAPTER XIII

THE BINOMIAL FORMULA

221. In Chapter II the following products were obtained:

$$(a+b)^2 = a^2 + 2\,ab + b^2.$$

$$(a+b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3.$$

$$(a+b)^4 = a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4.$$

$$(a+b)^5 = a^5 + 5\,a^4b + 10\,a^3b^2 + 10\,a^2b^3 + 5\,ab^4 + b^5.$$

By a study of these the following facts may be observed:

1. Each product has one term more than the number of units in the exponent of the binomial.

2. The exponent of a in the *first* term is the same as the exponent of the binomial, and diminishes by unity in each *succeeding* term.

The exponent of b in the *last* term is the same as the exponent of the binomial, and diminishes by unity in each *preceding* term.

3. The sum of the exponents in each term is equal to the exponent of the binomial.

4. The coefficient of the first term is unity; of the second term, the same as the exponent of the binomial; and the coefficient of any other term may be found by multiplying the coefficient of the next preceding term by the exponent of a in that term and dividing this product by a number one greater than the exponent of b in that term.

5. The coefficients of any pair of terms equally distant from the ends are equal.

Statements 2 and 4 form a rule for writing out any power of a binomial up to the fifth. Let us find $(a+b)^6$.

Multiplying $(a+b)^5$ by $a+b$, we have

$$\begin{aligned}(a+b)^5(a+b) &= a^6 + 5a^5b + 10a^4b^2 + 10a^3b^3 + 5a^2b^4 + ab^5\\ &\qquad\quad\; a^5b + 5a^4b^2 + 10a^3b^3 + 10a^2b^4 + 5ab^5 + b^6\end{aligned}$$

Hence $(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

From this it is seen that the rule holds also for $(a+b)^6$.

PROOF BY MATHEMATICAL INDUCTION

222. A proof that the above rule holds for *all positive integral powers* of a binomial may be made as follows:

First step. Write out the product as it *would be* for the nth power on the supposition that the rule holds.

Then the first term would be a^n and the last term b^n. The second terms from the ends would be $na^{n-1}b$ and nab^{n-1}. The third terms from the ends would be $\frac{n(n-1)}{1\cdot 2}a^{n-2}b^2$ and $\frac{n(n-1)}{1\cdot 2}a^2b^{n-2}$. The fourth terms from the ends would be

$$\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 \text{ and } \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^3b^{n-3},$$

and so on, giving by the hypothesis,

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \cdots + \frac{n(n-1)}{1\cdot 2}a^2b^{n-2} + nab^{n-1} + b^n.$$

Second step. Multiply this expression by $a+b$ and see if the result can be so arranged as to conform to the same rule.

Then, $\qquad (a+b)^n(a+b)$

$$\begin{aligned}&= a^{n+1} + na^nb + \frac{n(n-1)}{2}a^{n-1}b^2 + \cdots \qquad\qquad + na^2b^{n-1} + ab^n\\ &\qquad\quad a^nb \qquad\quad + na^{n-1}b^2 + \cdots + \frac{n(n-1)}{1\cdot 2}a^2b^{n-1} + nab^n + b^{n+1}.\end{aligned}$$

Hence adding,

$$\begin{aligned}(a+b)^{n+1} &= a^{n+1} + (n+1)a^nb + \left[\frac{n(n-1)}{1\cdot 2} + n\right]a^{n-1}b^2 + \cdots\\ &\qquad + \left[n + \frac{n(n-1)}{1\cdot 2}\right]a^2b^{n-1} + (n+1)ab^n + b^{n+1}.\end{aligned}$$

Combining the terms in brackets, we have,

$$\begin{aligned}(a+b)^{n+1} &= a^{n+1} + (n+1)a^nb + \frac{(n+1)n}{1\cdot 2}a^{n-1}b^2 + \cdots\\ &\qquad + \frac{(n+1)n}{1\cdot 2}a^2b^{n-1} + (n+1)ab^n + b^{n+1}.\end{aligned}$$

The last result shows that the rule holds for $(a+b)^{n+1}$ *if it holds for* $(a+b)^n$. That is, *if the rule holds for any positive integral exponent, it holds for the next higher integer.*

Third step. It was found above by *actual multiplication* that the rule does hold for $(a+b)^6$. Hence by the above argument we know that the rule holds for $(a+b)^7$.

Moreover, since we now know that the rule holds for $(a+b)^7$, we conclude by the same argument that it holds for $(a+b)^8$, and if for $(a+b)^8$, then for $(a+b)^9$, and so on.

Since this process of extending to higher powers can be carried on indefinitely, we conclude that the five statements in § 221 hold for all positive integral powers of a binomial.

The essence of this proof by **mathematical induction** consists in applying the *supposed* rule to the nth power and finding that the rule does hold for the $(n+1)$th power *if it holds for the nth power.*

223. **The general term.** According to the rule now known to hold for any positive integral exponent, we may write as many terms of the expansion of $(a+b)^n$ as may be desired, thus:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2$$

$$+\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4}a^{n-4}b^4 + \cdots. \qquad \text{I}$$

From this result, called the **binomial formula**, we see:

(1) The exponent of b in any term is one less than the number of that term, and the exponent of a is n minus the exponent of b. Hence the exponent of b in the $(k+1)$st term is k, and that of a is $n-k$.

(2) In the coefficient of any term the last factor in the denominator is the same as the exponent of b in that term, and the last factor in the numerator is one more than the exponent of a.

Hence the $(k+1)$st term, which is called the **general term** is

$$\frac{n(n-1)(n-2)(n-3)\cdots(n-k+1)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots k}a^{n-k}b^k. \qquad \text{II}$$

224. The process of writing out the power of a binomial is called **expanding the binomial**, and the result is called the **expansion of the binomial.**

Ex. 1. Expand $(x-y)^4$.

In this case $a = x$, $b = -y$, $n = 4$.

Hence substituting in formula I,

$$(x-y)^4 = x^4 + 4x^3(-y) + \frac{4(4-1)}{2}x^2(-y)^2 + \frac{4(4-1)(4-2)}{2\cdot 3}x(-y)^3$$
$$+\frac{4(4-1)(4-2)(4-3)}{2\cdot 3\cdot 4}(-y)^4 \qquad (1)$$

$$= x^4 - 4x^3y + \frac{4\cdot 3}{2}x^2y^2 - \frac{4\cdot 3\cdot 2}{2\cdot 3}xy^3 + \frac{4\cdot 3\cdot 2\cdot 1}{2\cdot 3\cdot 4}y^4. \qquad (2)$$

Hence $(x-y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$. $\qquad$ (3)

Notice that this is precisely the same as the expansion of $(x+y)^4$ except that every other term beginning with the second is *negative*.

Ex. 2. Expand $(1-2y)^5$.

Here $a = 1$, $b = -2y$, $n = 5$.

Since the coefficients in the expansion of $(a+b)^5$ are 1, 5, 10, 10, 5, 1, we write at once,

$$(1-2y)^5 = 1^5 + 5\cdot 1^4\cdot(-2y) + 10\cdot 1^3\cdot(-2y)^2$$
$$+ 10\cdot 1^2\cdot(-2y)^3 + 5\cdot 1\cdot(-2y)^4 + (-2y)^5$$
$$= 1 - 10y + 40y^2 - 80y^3 + 80y^4 - 32y^5.$$

Ex. 3. Expand $\left(\frac{1}{x}+\frac{y}{3}\right)^5$.

Remembering the coefficients just given, we write at once,

$$\left(\frac{1}{x}+\frac{y}{3}\right)^5 = \left(\frac{1}{x}\right)^5 + 5\left(\frac{1}{x}\right)^4\left(\frac{y}{3}\right) + 10\left(\frac{1}{x}\right)^3\left(\frac{y}{3}\right)^2 + 10\left(\frac{1}{x}\right)^2\left(\frac{y}{3}\right)^3$$
$$+5\left(\frac{1}{x}\right)\left(\frac{y}{3}\right)^4 + \left(\frac{y}{3}\right)^5\left(\frac{1}{x}+\frac{y}{3}\right)^5$$

$$= \frac{1}{x^5} + \frac{5}{3}\frac{y}{x^4} + \frac{10}{9}\frac{y^2}{x^3} + \frac{10}{27}\frac{y^3}{x^2} + \frac{5}{81}\frac{y^4}{x} + \frac{y^5}{243}.$$

In a similar manner any positive integral power of a binomial may be written.

Ex. 4. Write the *sixth term* in the expansion of $(x-2\,y)^{10}$ without computing any other term.

From II, § 223, we know the $(k+1)$st term for the nth power of $a+b$, namely,
$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{2\cdot 3\cdot 4\cdots k}a^{n-k}b^{k}.$$

In this case $a=x$, $b=-2\,y$, $n=10$, $k+1=6$. Hence $k=5$.

Substituting these particular values, we have
$$\frac{10(10-1)(10-2)\cdots(10-5+1)}{2\cdot 3\cdot 4\cdot 5}x^{10-5}(-2\,y)^5$$
$$=\frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{2\cdot 3\cdot 4\cdot 5}x^5(-32\,y^5)$$
$$=-32\cdot 252\,x^5y^5=-8064\,x^5y^5.$$

EXERCISES

1. Make a list of the coefficients for each power of a binomial from the 2d to the 10th.

Expand the following:

2. $(x-y)^3$.

3. $(2\,x+3)^3$.

4. $(3\,x+2\,y)^4$.

5. $(3+y)^5$.

6. $(x^3+y)^6$.

7. $(x-y^3)^6$.

8. $(x^3-y^2)^7$.

9. $(x^{\frac{1}{2}}-y^{\frac{1}{2}})^4$.

10. $(x^{-1}+y^{-2})^5$.

11. $(a-b)^8$.

12. $(x+y)^9$.

13. $(m-n)^{10}$.

14. $(r^{\frac{1}{2}}+s^2)^4$.

15. $(c^{-2}-d^{-\frac{1}{2}})^5$.

16. $(\sqrt{a}-\sqrt{b})^6$.

17. $\left(\frac{x^2}{y}-\frac{y^2}{x}\right)^3$.

18. $\left(\frac{2\,x}{y^2}-y\sqrt{x}\right)^3$.

19. $\left(\frac{\sqrt{m}}{\sqrt[3]{n^2}}+\sqrt{\frac{y}{n}}\right)^4$.

20. $\left(\frac{c\sqrt[3]{c}}{\sqrt[5]{d^4}}-\frac{\sqrt[3]{d}}{c}\right)^7$.

21. $(2\,a^2x^{-2}-3\,by^{-3})^4$.

22. $(3\,xy^{-3}-x^{-3}y)^8$.

In each of the following find the term called for without finding any other term:

23. The 5th term of $(a+b)^{12}$.

24. The 7th term of $(3x-2y)^{11}$.

25. The 6th term of $(\sqrt{x}-\sqrt{y})^{10}$.

26. The 9th term of $(x-y)^{25}$.

27. The 8th term of $(\frac{1}{2}m-\frac{1}{3}n)^{18}$.

28. The 7th term of $(a^2b-ab^2)^{20}$.

29. The 6th term of $(a-a^{-1})^{25}$.

30. The 11th term of $(x^2y-x^{-2}y^{-1})^{50}$.

31. The 5th term from each end of the expansion of $(a-b)^{20}$.

32. The 7th term from each end of $(a\sqrt{a}-b\sqrt{b})^{21}$.

33. Which term, counting from the beginning, has the same coefficient as the 7th term of $(a+b)^{10}$? Verify by finding both coefficients. How do the exponents differ in these terms?

34. What other term has the same coefficient as the 19th term of $(a+b)^{24}$? How do the exponents differ? Find in the shortest way the 21st term of $(a+b)^{25}$.

35. Find the 87th term of $(a+b)^{90}$.

36. Find the 53d term of $(a^{\frac{1}{2}}-b^{\frac{1}{2}})^{56}$.

37. What other term has the same coefficient as the 5th term in the expansion of $(x+y)^{19}$?

38. Expand $[(a+b)+c]^3$ by the binomial formula.

39. Expand $[1+(2x+3y)]^4$ by the binomial formula.

40. Expand $(2x-3y+4z)^3$ by the binomial formula.

41. Write the $(k+1)$st term of $(a+b)^n$. Write the $(n+1)$st term of $(a+b)^n$. Show that the next and also all succeeding terms after the $(n+1)$st term have zero coefficients, thus proving that there are exactly $n+1$ terms in the expansion.

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